1973

Analysis of continuous curvilinear structures by infinite matrix series methods

Trinh-Ngoc Rang

ANALYSIS OF CONTINUOUS CURVILINEAR STRUCTURES

BY INFINITE MATRIX SERIES METHODS

BY

TRINH NGOC RANG, 1940

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI-ROLLA

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

CIVIL ENGINEERING

May, 1973

John L Best
Advisor

Joseph H Senne

Oktay Ural

Wei-Wen Yu

C. Y. Ho

K. G. Muhlbauer

Antonio [illegible]

ABSTRACT

The objective of this dissertation is to present two methods of structural analysis for continuous curvilinear frames by using infinite matrix series as an extension of the well-known moment-distribution method. The sum of all unbalanced moments and thrusts relaxed at joints of a continuous system can be expressed in a compact and exact mathematical expression, which is in terms of the sum of a convergent infinite matrix series. With these unbalanced forces at joints, the final support forces may be calculated in a single stage of distribution and carry-over. The convergence of the balancing process is also demonstrated. Approximate results can be obtained by taking the partial sums of the infinite matrix series; in these cases, the errors that may be committed in stopping at any stage of balancing can be estimated.

Flexibilities, stiffnesses, restraints along with distribution, carry-over and transmission factors of segmental arches are derived in general matrix forms.

Stiffness, carry-over and thrust-induction factors as well as fixed-end reaction coefficients (moment, thrust and shear) of segmental arches of different symmetrical types are derived, graphed and tabulated for use.

Numerical examples are given to illustrate the procedure; computer programs are developed to effectively solve complex structures and an experimental model was built and tested by using the Beggs deformeter to correlate the results.

ACKNOWLEDGMENTS

The author wishes to express his thanks to his major advisor, Dr. J.L. Best, Professor of Civil Engineering, for his encouragement and continued guidance during the course of this research work. The author also wishes to acknowledge Professor A.J. Celis for his lectures which inspired the author to take up this topic. Thanks are extended to members of the Graduate Committee: Dr. J.H. Senne, Dr. O. Ural, Dr. W.W. Yu of the Civil Engineering Department, Dr. B.E. Gillett, Dr. C.Y. Ho of the Computer Science Department and Dr. K.C. Muhlbauer of the Engineering Mechanics Department.

Acknowledgment is given to the National Institute of Technology Saigon and the Agency for International Development of the U.S. Department of State, whose sponsorship permitted the author to complete his graduate studies. This financial assistance is gratefully acknowledged.

Finally, special appreciation must be offered to his wife, Kim-Yen and his daughters Anh-Thu and Huong-Thu who have been patient through some of their longest years.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF SYMBOLS

x, y	horizontal and vertical coordinates
ds	elementary length of a curved member
Δs	curved length of a finite section
u, v, θ	horizontal, vertical and rotational displacements
H	horizontal force or thrust
V	vertical force or shear
M	bending moment
h	horizontal force due to unit displacement
m	bending moment due to unit displacement
A	area of the cross-section
I	moment of inertia of the cross-section
I_o	moment of inertia at the crown of the arch
E	modulus of elasticity
L	arch span
r	arch rise
r/L	rise to span ratio
R	radius of circular arch
w	uniformly distributed load per unit length
P	concentrated load
X	location of concentrated load along x-axis
α	inclination of the tangent to the arch with respect to the horizontal
ω	angle which locates the concentrated load on circular arch
β	one-half the total subtended angle
γ	thrust-induction factor

γ'	modified thrust-induction factor
ψ	flexural parameter
λ	eigenvalue
n	number of interior joints of a continuous system
k	number of times of joint release
p	number of cycles of balancing

In the following quantities, p = 1, 2, 3 and q = 1, 2, 3

1 refers to thrust or horizontal direction
2 refers to shear or vertical direction
3 refers to moment or rotation

fpq	flexibility factor
kpq	stiffness factor
kpq'	modified stiffness factor
spq	stiffness coefficient
rpq	restraint factor
rpq'	modified restraint factor
dpq	distribution factor
cpq	carry-over factor
tpq	transmission factor

Subscripts

1, 2, i	means that the quantity occurs at 1, 2, i
Ri	means that the quantity occurs at the right side of joint i
Li	means that the quantity occurs at the left side of joint i
Ci	means that the quantity occurs at the column side of joint i
ij, AB	means for member ij, AB
c	means that the quantity refers to the column

Superscripts

s	means that the quantity refers to determinate base structure
E	means that the quantity refers to the external loads
f	means fixed-end quantity
u	means unbalanced quantity
d	means distributed quantity
c	means carry-over quantity
r	means that the quantity is due to rotation
e	means end quantity, i.e., quantity occurs at the end of each loading condition

Matrix and vector notation

$[f_A]$	flexibility matrix at A of a segmental arch AB
$[k_A]$	stiffness matrix at A of a segmental arch AB
$[k_A']$	modified stiffness matrix at A
$[r_B]$	restraint matrix at B of a segmental arch AB
$[r_B']$	modified restraint matrix at B
$[\sum k_i]$	matrix of sum of stiffness factors of various members framing into the joint i
$[d_i]$	distribution factor matrix for a segmental arch framing into the joint i
$[c_i]$	carry-over factor matrix for a segmental arch framing into the joint i
$[t_i]$	transmission factor matrix for a segmental arch framing into the joint i
$[D_{Ri}]$	distribution factor matrix for the right side of joints of a continuous system
$[D_{Li}]$	distribution factor matrix for the left side of joints of a

continuous system

$[D_{Ci}]$ distribution factor matrix for the column side of joints of a continuous system

$[T_i]$ transmission matrix of a continuous system

$[T'_{Ri}]$ transpose of the transmission matrix for the right side of joints of a continuous system

$[T'_{Li}]$ transpose of the transmission matrix for the left side of joints of a continuous system

$\{F_i^f\}$ generalized fixed-end force vector at i

$\{F_i^d\}$ generalized distributed force vector at i

$\{F_i^E\}$ generalized force vector at joint i due to external loads

$\{F_i^u\}$ generalized unbalanced force vector at joint i

$\{F_i\}$ generalized force vector

$\{\delta_i\}$ generalized displacement vector

$\{u_i\}$ horizontal displacement vector

$\{M_i^u\}_o$ initial unbalanced moment vector at joints due to external loads

$\{H_i^u\}_o$ initial unbalanced thrust vector at joints due to external loads

$[m_{ij}^u]_o$ initial unbalanced moment matrix due to unit displacements at joints

$[h_{ij}^u]_o$ initial unbalanced thrust matrix due to unit displacements at joints

$\{F_i^u\}_o$ initial generalized unbalanced force vector at joints

$\{M_i^u\}_k$ unbalanced moment vector relaxed at joints at the end of the k-th release in the balancing process

$\{F_i^u\}_k$ generalized unbalanced force vector relaxed at joints at the end of the k-th release in the balancing process

$\{M_i^u\}_s$	vector of sum of all moments relaxed at joints due to external loads
$[m_{ij}^u]_s$	matrix of sum of all moments relaxed at joints due to unit displacements
$\{F_i^u\}_s$	vector sum of all generalized forces relaxed at joints
$\{M_i^u\}_t$	total unbalanced moment vector at joints due to external loads
$[m_{ij}^u]_t$	total unbalanced moment matrix due to unit displacements at joints
$\{F_i^u\}_t$	total generalized unbalanced force vector at joints
$[Q] = [I] - [T']$	
$[S]$	matrix of sum of the infinite matrix in $[T']$ or inverse matrix of $[Q]$
$[S_k]$	matrix of partial sum of the infinite matrix series in $[T']$ up to the k-th power of $[T']$
$[E_k]$	error matrix at the end of the k-th stage
$[E_t]$	total error matrix
$\{e\}_k$	error vector at the end of the k-th stage
$\|T\|_1$	first norm of the matrix $[T]$
$\|T\|_2$	second norm of the matrix $[T]$
$\|T\|_3$	third norm of the matrix $[T]$
$\|E_k\|_3$	third norm of the error matrix $[E_k]$ at the end of the k-th stage

Chapter 1

INTRODUCTION

1.1 Continuous Curvilinear Structure

Curvilinear structures may occur in frames which may be continuous with their piers or columns and also with girders or with other arches. Examples of the use of continuous curvilinear structures are numerous: multiple arch bridges resting on slender piers, buildings with curved girders, multi-bay portals with curved roof systems, multiple viaducts, curvilinear grid systems and others. Curved members used in combination with girders offer interesting structural and architectural design features.

As would be expected, the flexural properties of curved members are more difficult to evaluate than those for straight prismatic members, and so the expressions involved in developing these properties for curvilinear structures are rather more complicated.

1.2 Literature Review

Several methods are available for the analysis of continuous curvilinear structures. The moment-distribution analysis of continuous frames was first presented by H. Cross[1]. This method was almost immediately modified to apply to continuous arches by D.E. Larson[2]. Since then the extension of the conception of balancing moments and thrusts to the problem of the analysis of continuous arches on elastic piers has been proposed by many investigators. H. Cross and N.D. Morgan[3] used a procedure in which thrusts are distributed at a so-called neutral point of each joint. Other solutions have been proposed by A. Hrennikoff[4], L.C. Maugh[5], V.A. Morgan[6], J. Michalos and D.D. Girton[7], W.E. Riley[8], J.J Tuma, K.S. Havner and F. Hedges[9] and others. Most of these solutions follow the principle of moment-distribution with restraining forces to prevent sidesway and a subsequent number of distributions to remove the restraints. Some modifications to speed the convergence were employed by these investigators. Solutions based on the use of slope-deflection equations for continuous arches were first proposed by H.P. Hjort[10] and then by K.T. Fowler[11], J.I. Parcel and Moorman[12], E. Lightfoot[13], T.M. Wang and D.C. Church[14]. All of these approaches require the solution of simultaneous equations to obtain the slopes and displacements at joints of support. Moments and thrusts are then obtained by back-substitution. L.A. Beaufoy and L.A.S. Diwan[15], E. Lightfoot[16], F.P. Wiesinger, S.L. Lee and D. L. Guell[17] have presented solutions that do not require the use of either restraining forces or the solution of simultaneous equations.

The infinite series approach is an analytical procedure based on the philosophy of moment-distribution. The moments successively carried over to a certain joint of a continuous system arrange themselves in a convergent infinite series. Hence, the sum of its terms is readily given by a summation formula.

The concept of infinite series solutions to continuous beams and frames has been developed by H. Cross[18] and others[19], but the extent of their investigations were quite restricted. A. Pauw[20] defined a new structural parameter called "sequence-summation factors" which may be used in a systematic calculation of over-relaxation factors for moment-distribution processes. These sequence-summation factors are computed directly from the distribution and carry-over factors used in the moment-distribution method. Yoshimura[21] developed a method which deals with the geometric series of carry-over moments for the analysis of continuous beams. The procedure is based on relations existing between the adjacent carry-over moments given by the rules of distribution and carry-over to establish a set of simultaneous equations from which the required sums of carry-over moments are deduced. J.J. Tuma and others[22,23,9] derived expressions for joint moments by using an infinite series approach. These moments are functions of a new distribution factor and fixed-end moments. The new distribution factor consists of two classes of infinite series: basic series for the case of two or three span beams and carry-over series for the case of five and more span beams. These series consist of an infinite number of terms converging to zero. Each term of the carry-over series is another infinite series.

1.3 Scope of Investigation

The objective of this dissertation is to present a general analytical solution of continuous curvilinear structures by infinite matrix series methods. These methods are based on the philosophy of moment-distribution and the matrix methods of structural analysis. This assumes that moments and forces at joints of a continuous system can be expressed in exact and compact mathematical expressions which are in terms of the sum of an infinite matrix series.

Chapter 2 deals with the properties of segmental arches.

Flexibility, stiffness, restraint factors as well as distribution, carry-over and transmission factors of segmental arches framing into a joint are expressed in matrix forms. The general deformation-force matrix equation is also derived.

Stiffnesses, carry-over and thrust-induction factors along with fixed-end reaction coefficients due to different loading conditions for various types of symmetrical arches are graphed and tabulated for use in the analysis of continuous arches.

Chapter 3 presents two proposed methods for the analysis of continuous curvilinear structures. These are to be defined as the restrained infinite matrix series method and the generalized infinite matrix series method.

In the restrained infinite matrix series method, unbalanced moments at joints, without joint translation, can be expressed in terms of an infinite matrix series sum. End moments due to external loads and unit displacements at joints can be computed from these unbalanced moments. The actual horizontal displacements of joints may be obtained from the solution of equilibrium equations of unbalanced thrusts at these joints. The final moments and thrusts are then obtained by back-substitution.

In the generalized infinite matrix series method, joints are released one by one which allows both translation and rotation simultaneously and the generalized unbalanced forces are functions of an infinite matrix series. The final moments and thrusts are obtained directly from these unbalanced generalized forces by a single stage of distribution and carry-over.

Approximate results can be obtained by using various partial sums of these infinite matrix series. Errors that may be committed in stopping at any stage of balancing process are also evaluated and the convergence of the transmission matrix is demonstrated.

Chapter 4 presents the application of analysis. A mathematical model is introduced to illustrate the two proposed methods. Numerical examples of approximations and error computations are given to justify the theory. Computer programs are written to effectively solve complex structures. Numerical examples for the computer solution are chosen to show the validity of the methods. Finally, an experimental model was built and tested by using the Beggs deformeter to correlate the results.

Chapter 2

SEGMENTAL ARCHES

2.1 Flexibilities

A flexibility factor, fpq, of a segmental arch AB is defined as the displacement of point A in the p direction due to a unit action at that point in the q direction, all other points being unloaded. Apparently, the flexibility factor constitutes a relationship between deformation and force. Applying the principle of superposition, the deformation at any point of a system caused by a set of forces may be expressed in terms of flexibility factors.

Consider a fixed-end curved member AB of variable cross-section as shown in Fig. 1.a. It is seen that the structure is statically indeterminate to the third degree. If the origin of the cartesian axes were at point A and if point A were released, the structure becomes statically determinate as shown in Fig. 1.b and is called the base structure. The three redundants at end A, namely, the thrust H_A, the shear V_A and the bending moment M_A, may be taken to act there to prevent linear or angular displacements. The positive sign convention is as shown in Fig. 1.c.

Suppose the loading on the member AB produces a bending moment M and a thrust H at any point (x,y) on the member. Then,

$$M = M^s + M_A - H_A y + V_A x \qquad (2.1.a)$$

$$H = H^s + H_A \frac{dx}{ds} + V_A \frac{dy}{ds} \qquad (2.1.b)$$

where M^s and H^s are moment and thrust due to external loads on the determinate base structure.

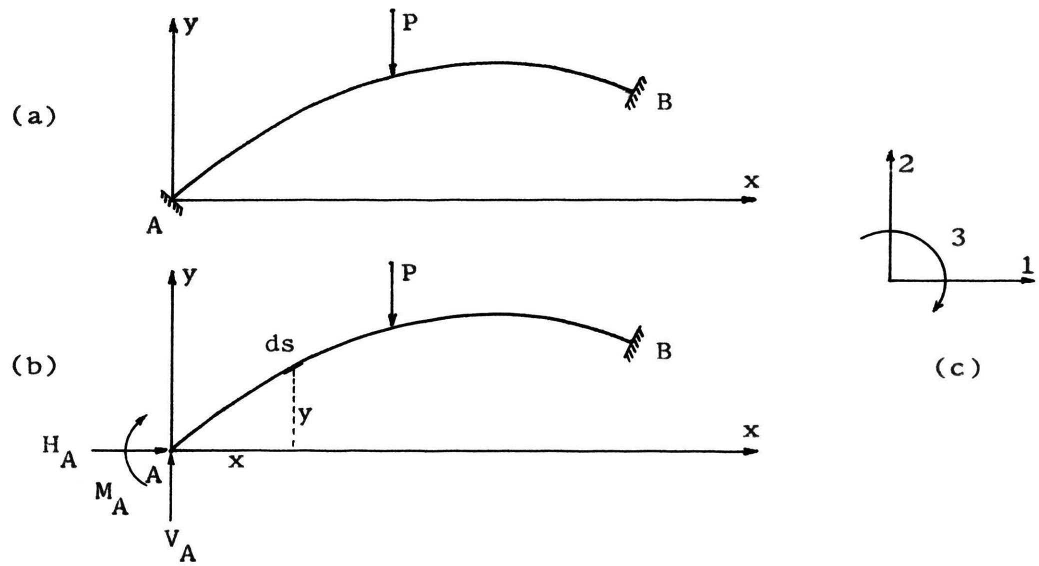


Fig. 1. Segmental Arch

Eqs. (2.1.a) and (2.1.b) yield,

$$\frac{\partial M}{\partial H_A} = -y, \quad \frac{\partial M}{\partial V_A} = x, \quad \frac{\partial M}{\partial M_A} = 1 \tag{2.2.a}$$

$$\frac{\partial H}{\partial H_A} = \frac{dx}{ds}, \quad \frac{\partial H}{\partial V_A} = \frac{dy}{ds}, \quad \frac{\partial H}{\partial M_A} = 0 \tag{2.2.b}$$

Ignoring shear strains and using Castigliano's second theorem, the displacements at end A can be written as follows:

$$u_A = \int \frac{M \frac{\partial M}{\partial H_A}}{EI} ds + \int \frac{H \frac{\partial H}{\partial H_A}}{EA} ds \tag{2.3.a}$$

$$v_A = \int \frac{M \frac{\partial M}{\partial V_A}}{EI} ds + \int \frac{H \frac{\partial H}{\partial V_A}}{EA} ds \tag{2.3.b}$$

$$\theta_A = \int \frac{M \frac{\partial M}{\partial M_A}}{EI} ds + \int \frac{H \frac{\partial H}{\partial M_A}}{EA} ds \qquad (2.3.c)$$

Substituting values of Eqs. (2.2.a) and (2.2.b) into Eqs. (2.3), three simultaneous equations in H_A, V_A and M_A can be found from the conditions that the displacements u_A, v_A and the rotation θ_A at end A are zero.

$$u_A = 0 = u_A^s + f11_A H_A + f12_A V_A + f13_A M_A \qquad (2.4.a)$$

$$v_A = 0 = v_A^s + f21_A H_A + f22_A V_A + f23_A M_A \qquad (2.4.b)$$

$$\theta_A = 0 = \theta_A^s + f31_A H_A + f32_A V_A + f33_A M_A \qquad (2.4.c)$$

in which, u_A^s, v_A^s and θ_A^s are horizontal, vertical and rotational displacements respectively, due to external loads on the determinate base structure. Coefficients and constants of Eqs. (2.4) may be expressed as follows:

$$u_A^s = \int \frac{M^s}{EI} (-y)\, ds + \int \frac{H^s}{EA} \left(\frac{dx}{ds}\right) ds \qquad (2.5.a)$$

$$v_A^s = \int \frac{M^s}{EI} x\, ds + \int \frac{H^s}{EA} \left(\frac{dy}{ds}\right) ds \qquad (2.5.b)$$

$$\theta_A^s = \int \frac{M^s}{EI} ds \qquad (2.5.c)$$

$$f11_A = \int \frac{y^2}{EI} ds + \int \frac{1}{EA} \left(\frac{dx}{ds}\right)^2 ds \qquad (2.6.a)$$

$$f12_A = \int - \frac{xy}{EI} ds + \int \frac{1}{EA} \left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) ds \qquad (2.6.b)$$

$$f13_A = \int \frac{-y}{EI} ds \qquad (2.6.c)$$

$$f21_A = \int -\frac{xy}{EI}\, ds + \int \frac{1}{EA}\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) ds \qquad (2.6.d)$$

$$f22_A = \int \frac{x^2}{EI}\, ds + \int \frac{1}{EA}\left(\frac{dy}{ds}\right)^2 ds \qquad (2.6.e)$$

$$f23_A = \int \frac{x}{EI}\, ds \qquad (2.6.f)$$

$$f31_A = \int \frac{-y}{EI}\, ds \qquad (2.6.g)$$

$$f32_A = \int \frac{x}{EI}\, ds \qquad (2.6.h)$$

$$f33_A = \int \frac{1}{EI}\, ds \qquad (2.6.i)$$

These expressions include 14 different integrals, five of which involve external loads.

Flexibilities can be expressed in matrix form as follows:

$$[f_A] = \begin{bmatrix} f11_A & f12_A & f13_A \\ f21_A & f22_A & f23_A \\ f31_A & f32_A & f33_A \end{bmatrix} \qquad (2.7)$$

$[f_A]$ is called flexibility matrix at end A of the segmental arch AB.

From Betti 's theorem, it is found that

$$f21_A = f12_A, \quad f31_A = f13_A, \quad f32_A = f23_A \qquad (2.8)$$

2.2 Stiffnesses

A stiffness factor, kpq_A, of a curved member AB is defined as the force required to act in the p direction to produce a unit displacement in the q direction at the end A of the member, all other points being fixed. In the same manner as exhibited by the flexibility factor, the stiffness factor constitutes a relationship between force and displacement. Applying the principle of superposition, the force component of a system may be expressed in terms of a set of prescribed displacements.

Stiffnesses depend on the geometry, the structural properties of a curved member, axial effects and end restraint conditions.

Stiffnesses at end A of a curved member AB can be written in matrix form as follows:

$$[k_A] = \begin{bmatrix} k11_A & k12_A & k13_A \\ k21_A & k22_A & k23_A \\ k31_A & k32_A & k33_A \end{bmatrix} \tag{2.9}$$

Referring to Fig. 2, elements of the stiffness matrix are defined as follows:

k11 = thrust stiffness

k12 = thrust-shear stiffness

k13 = thrust-moment stiffness

k21 = shear-thrust stiffness

k22 = shear stiffness

k23 = shear-moment stiffness

k31 = moment-thrust stiffness

k32 = moment-shear stiffness

k33 = moment stiffness

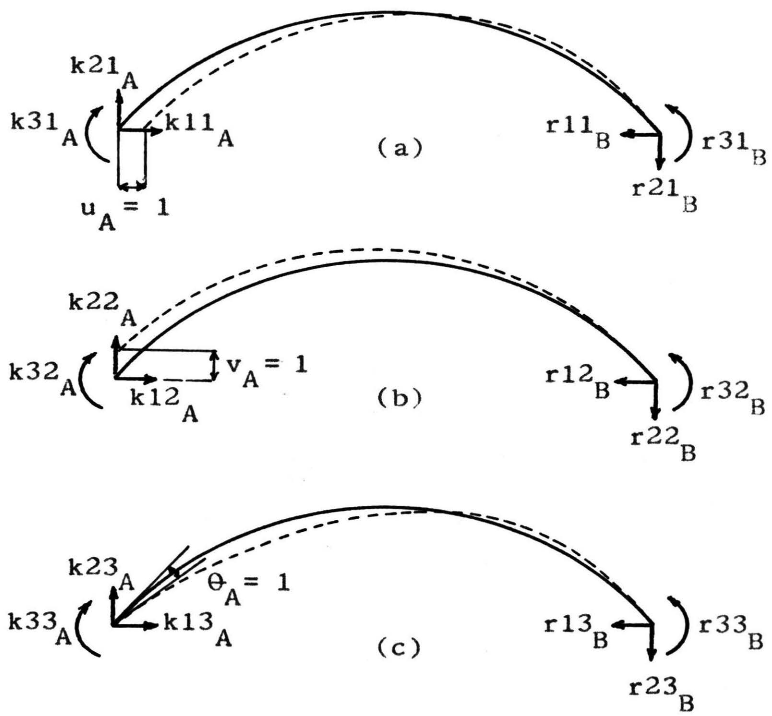


Fig. 2. Definition of Stiffness and Restraint Factors

Stiffnesses can be derived by method of column analogy[16,24] or by displacement method[15]. With a curved member, however, it is found to be much easier to obtain stiffnesses by inverting the flexibility matrix.

Thus,

$$[k_A] = [f_A]^{-1} \tag{2.10}$$

There are thus nine separate stiffness factors, but three conjugate relations hold for stiffness factors deduced from Betti's theorem. These are

$$k21_A = k12_A, \quad k31_A = k13_A, \quad k32_A = k23_A \tag{2.11}$$

For a curved member symmetrical about the y-axis, it is found that $k12_A = k21_A = 0$; so the stiffness matrix at the end A of the symmetrical member AB may be written

$$[k_A] = \begin{bmatrix} k11_A & 0 & k13_A \\ 0 & k22_A & k23_A \\ k31_A & k32_A & k33_A \end{bmatrix} \qquad (2.12)$$

If the springing ends of the arch member undergo no vertical displacement, then the plane curved member considered in this dissertation has only two independent joint movements (horizontal and rotational displacements); the stiffness matrix becomes

$$[k_A] = \begin{bmatrix} k11_A & k13_A \\ k31_A & k33_A \end{bmatrix} \qquad (2.13)$$

2.3 Restraints

When a system of forces $[k_A]$ is applied at the end A of a curved member AB, a corresponding system of forces has to be exerted on the member AB at the end B to prevent displacements there. Thus, a restraint factor, rpq_B, at the end B of a member AB may be defined as the force required to act in the p direction, at the end B when a unit displacement is imposed in the q direction at the end A. The restraint matrix at the end B of the member AB is

$$[r_B] = \begin{bmatrix} r11_B & r12_B & r13_B \\ r21_B & r22_B & r23_B \\ r31_B & r32_B & r33_B \end{bmatrix} \qquad (2.14)$$

Elements of the restraint matrix are defined as follows:

r11 = thrust restraint

r12 = thrust-shear restraint

r13 = thrust-moment restraint

r21 = shear-thrust restraint

r22 = shear restraint

r23 = shear-moment restraint

r31 = moment-thrust restraint

r32 = moment-shear restraint

r33 = moment restraint

The restraint factors are directly related to the corresponding stiffness factors, since each set of factors must be in static equilibrium with the set of the other end of the member. Referring to Fig. 2, it is seen that the following equations apply for each corresponding set of stiffness and restraint factors:

$$\left.\begin{aligned} & k1j + r1j = 0 \\ & k2j + r2j = 0 \\ & k3j + r3j + r1j*(y_B - y_A) + r2j*(x_B - x_A) = 0 \end{aligned}\right\} \qquad (2.15)$$

where $j = 1, 2, 3$ and x_A, y_A and x_B, y_B are coordinates of end A and end B respectively. Thus the restraint factors can be easily obtained from the stiffness factors by means of Eqs. (2.15), which are static equilibrium equations for the member AB.

For a curved member symmetrical about the y-axis, it is evident that $r12_B = r21_B = 0$ and the restraint matrix at the end B of the member AB may be written

$$[r_B] = \begin{bmatrix} -k11_A & 0 & -k13_A \\ 0 & -k22_A & k23_A \\ -k31_A & -k32_A & c33_A k33_A \end{bmatrix} \tag{2.16}$$

in which $c33_A$ is called the moment carry-over factor at the end A of the segmental arch AB. Let $L = x_B - x_A$ = arch span, and it follows that

$$c33_A = \frac{-k33_A - k23_A L}{k23_A} \tag{2.17}$$

This term may be defined as the ratio of the induced moment at the far end which is fixed to the applied moment at the near end without linear displacements.

Eq. (2.16) shows that only the moment carry-over factors are required to enable the restraint matrix to be obtained from the stiffness matrix.

For the curved member in which the springings are only able to rotate and displace horizontally the restraint matrix becomes

$$[r_B] = \begin{bmatrix} -k11_A & -k13_A \\ -k31_A & -k33_A - k23_A L \end{bmatrix} \tag{2.18}$$

2.4. Deformation-Force Matrix Equation for Segmental Arch

The general deformation-force matrix equation for a curved member AB may be shown to be composed of sub-matrices, two of which are stiffness matrices at end A and end B and the other two are corresponding restraint matrices.

$$\begin{Bmatrix} H_A \\ V_A \\ M_A \\ \cdots \\ H_B \\ V_B \\ M_B \end{Bmatrix} = \left[\begin{array}{ccc:ccc} k11_A & k12_A & k13_A & r11_A & r12_A & r13_A \\ k21_A & k22_A & k23_A & r21_A & r22_A & r23_A \\ k31_A & k32_A & k33_A & r31_A & r32_A & r33_A \\ \hdashline r11_B & r12_B & r13_B & k11_B & k12_B & k13_B \\ r21_B & r22_B & r23_B & k21_B & k22_B & k33_B \\ r31_B & r32_B & r33_B & k31_B & k32_B & k33_B \end{array}\right] \begin{Bmatrix} u_A \\ v_A \\ \theta_A \\ \cdots \\ u_B \\ v_B \\ \theta_B \end{Bmatrix} \tag{2.19}$$

The deformation-force matrix equation can be written in condensed matrix form as follows:

$$\begin{Bmatrix} F_A \\ \cdots \\ F_B \end{Bmatrix} = \left[\begin{array}{c:c} k_A & r_A \\ \hdashline r_B & k_B \end{array}\right] \begin{Bmatrix} \delta_A \\ \cdots \\ \delta_B \end{Bmatrix} \tag{2.20}$$

In Eq. (2.20), $\{F_A\}$ and $\{F_B\}$ are redundant vectors at A and B. $[k_A]$, $[k_B]$ and $[r_A]$, $[r_B]$ are stiffness and restraint sub-matrices at A and B, and $\{\delta_A\}$ and $\{\delta_B\}$ are respective displacement vectors.

From the above relationship, the general form of slope-deflection equations for a segmental arch may be expressed as follows:

$$\left.\begin{aligned} \{F_A\} &= \{F_A^f\} + [k_A]\{\delta_A\} + [r_A]\{\delta_B\} \\ \{F_B\} &= \{F_B^f\} + [k_B]\{\delta_B\} + [r_B]\{\delta_B\} \end{aligned}\right\} \tag{2.21}$$

where $\{F_A^f\}$ and $\{F_B^f\}$ are fixed-end force vectors.

2.5 Flexibilities and Stiffnesses of Symmetrical Arches

Four types of symmetrical arches are studied in detail:

- Parabolic arch with secant variation in its I values,
- Uniform circular arch,
- Uniform parabolic arch,
- Uniform semi-elliptical arch.

The axial effect is also included for uniform configurations.

2.5.1 Parabolic Arch with Secant Variation in I

Fig. 3 shows the geometry of a parabolic segmental arch. Let L = arch span, r = arch rise and (x,y) = coordinates of a point on the center line of the arch. The equation representing this configuration is

$$y = 4r\left(-\frac{x^2}{L^2} + \frac{x}{L}\right) \tag{2.22}$$

Assuming that the moment of inertia at any section is related to that at the crown by $I = I_o \sec\alpha$, where I_o is the minimum moment of inertia at the crown and α is the inclination of the tangent to the arch with respect to the horizontal, then $ds = \sec\alpha dx$.

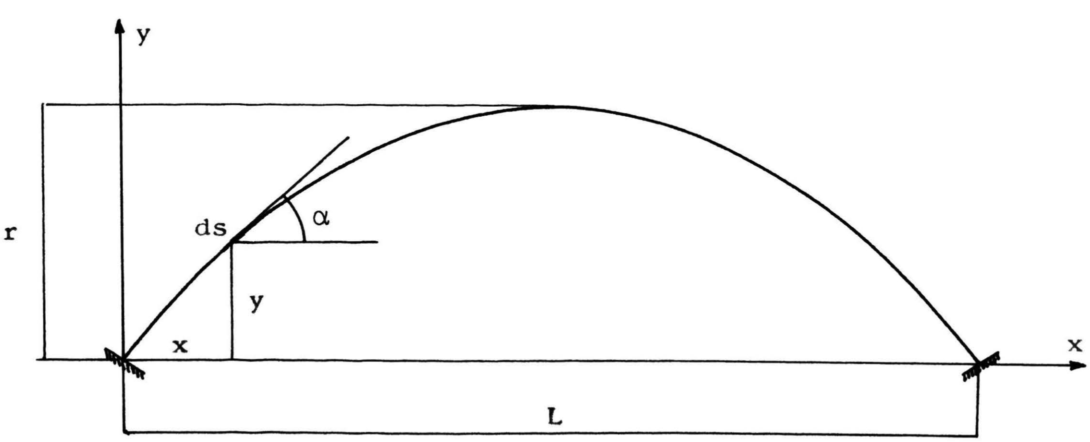


Fig. 3. Parabolic Segmental Arch

Elements of the flexibility matrix are then easily evaluated from Eqs. (2.6)

$$
\left.
\begin{aligned}
f11_A &= \frac{8r^2L}{15EI_o} \\
f12_A &= -\frac{rL^2}{3EI_o} \\
f13_A &= -\frac{2rL}{3EI_o} \\
f22_A &= \frac{L^3}{3EI_o} \\
f23_A &= \frac{L^2}{2EI_o} \\
f33_A &= \frac{L}{EI_o}
\end{aligned}
\right.
\qquad (2.23)
$$

2.5.2 Uniform Circular Arch

Consider the geometry of the circular segment as shown in Fig. 4 with the geometric constants defined as:

L = arch span

r = arch rise

R = radius of the arch

β = one-half the total subtended angle

If the origin of the cartesian axes is placed at springing end A, the coordinates of a point on the arch are as follows:

$$
\left.
\begin{aligned}
x &= R(\sin\beta - \sin\alpha) \\
y &= R(\cos\alpha - \cos\beta)
\end{aligned}
\right\}
\qquad (2.24)
$$

It follows that

$$\left.\begin{aligned} dx &= -R\cos\alpha d\alpha, \quad dy = -R\sin\alpha d\alpha, \quad ds = Rd\alpha \\ L &= 2R\sin\beta \\ r &= R(1 - \cos\beta) \end{aligned}\right\} \quad (2.25)$$

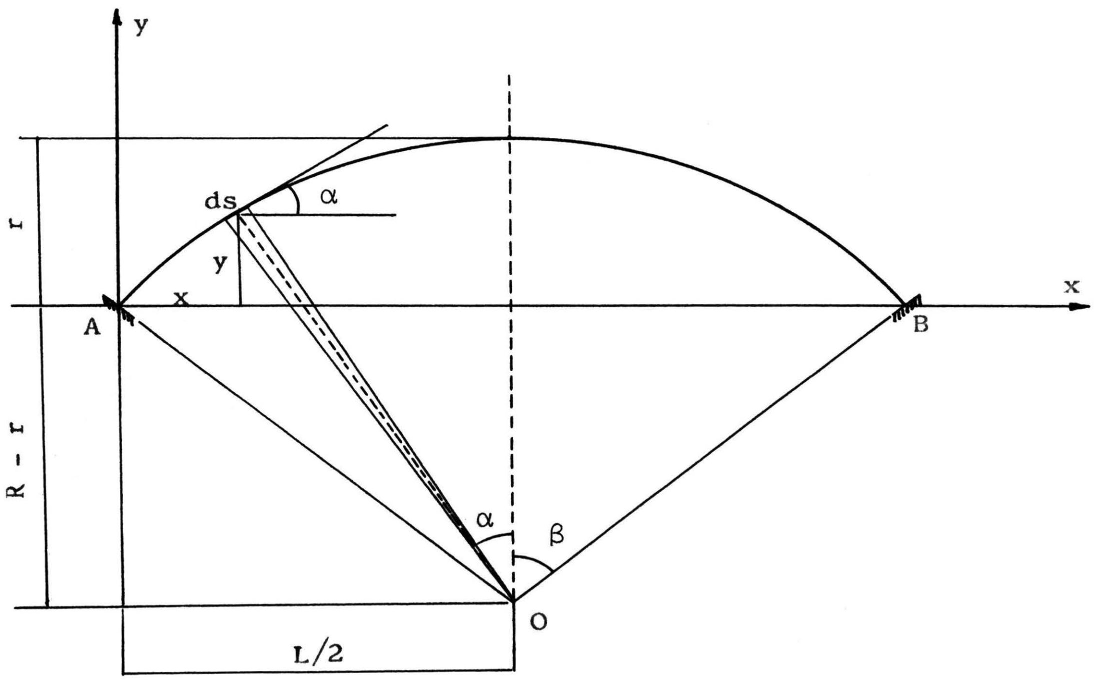


Fig. 4. Circular Segmental Arch

Elements of the flexibility matrix can be obtained by substituting Eqs. (2.24) and (2.25) into Eqs. (2.6); they are expressed as follows:

$$f11_A = \frac{R^3}{EI}\left(\beta - 1.5\sin 2\beta + 2\beta\cos^2\beta\right) + \frac{R}{EA}\left(\beta + 0.5\sin 2\beta\right)$$

$$f12_A = \frac{R^3}{EI}\left(\beta \sin 2\beta - 2\sin^2\beta\right)$$

$$f13_A = \frac{2R^2}{EI}\left(\beta\cos\beta - \sin\beta\right)$$

(2.26)

$$f22_A = \frac{R^3}{EI}\left(\beta + 2\beta\sin^2\beta - 0.5\sin 2\beta\right) + \frac{R}{EA}\left(\beta - 0.5\sin 2\beta\right)$$

$$f23_A = \frac{2R^2}{EI} \beta \sin\beta$$

$$f33_A = \frac{2R}{EI} \beta$$

2.5.3 Uniform Parabolic Arch

Expressions for the elements of the flexibility matrix will become impossible to evaluate by direct integration. These may be evaluated by Simpson's rule for numerical integration.

The equation to the center line of the parabolic arch together with their derivative functions are as follows:

$$\left.\begin{aligned} y &= 4r\left(-\frac{x^2}{L^2} + \frac{x}{L}\right) \\ \frac{dy}{dx} &= 4r\left(-\frac{2x}{L^2} + \frac{1}{L}\right) \\ \frac{ds}{dx} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\ \frac{dx}{ds} &= \frac{1}{ds/dx} \end{aligned}\right\} \qquad (2.27)$$

Substituting these equations into Eqs. (2.6) and using a subroutine for numerical integration the flexibility elements can be obtained with a good accuracy.

2.5.4 Uniform Semi-Elliptical Arch

Consider the semi-ellipse represented by Eq. (2.28) when the major axis lies on the x-coordinate and the origin is located at the left end as shown in Fig. 5.

$$y = \frac{2r}{L}\sqrt{Lx - x^2} \qquad (2.28)$$

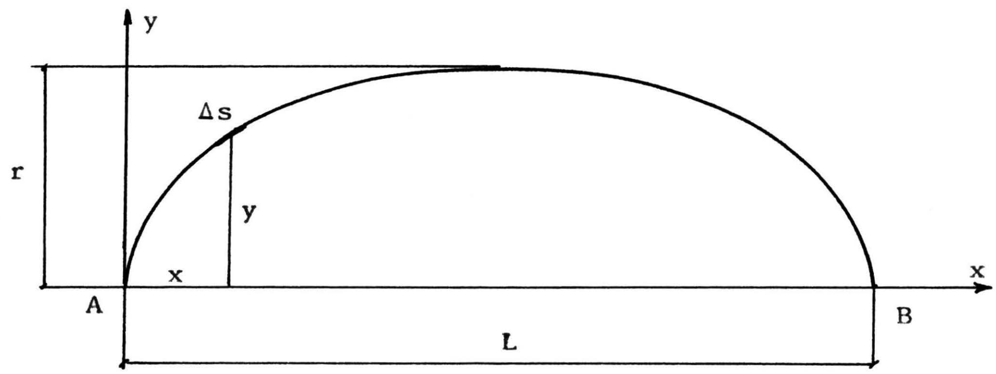


Fig. 5. Semi-Elliptical Arch

Since the expressions for the flexibilities cannot be integrated directly, a finite element technique with the aid of an electronic digital computer is used. For numerical solutions, the integration sign is replaced by the summation sign, and a finite number of Δs sections is used, where Δs is analogous to the differential ds in prior theoretical expressions.

Various elements of the flexibility matrix can be written as follows:

$$
\begin{aligned}
f11_A &= \sum y^2 \frac{\Delta s}{EI} + \sum \left(\frac{\Delta x}{\Delta s}\right)^2 * \frac{\Delta s}{EA} \\
f12_A &= \sum -xy \frac{\Delta s}{EI} + \sum \frac{\Delta x}{\Delta s}\frac{\Delta y}{\Delta s}\frac{\Delta s}{EA} \\
f13_A &= \sum -y \frac{\Delta s}{EI} \\
f22_A &= \sum x^2 \frac{\Delta s}{EI} + \sum \left(\frac{\Delta y}{\Delta s}\right)^2 * \frac{\Delta s}{EA} \\
f23_A &= \sum x \frac{\Delta s}{EI} \\
f33_A &= \sum \frac{\Delta s}{EI}
\end{aligned}
\tag{2.29}
$$

In the above finite summations, x and y are the coordinates to the center line of the arch.

Five thousand equal Δx spaces along horizontal axis has been used. This leads to unequal Δs lengths

$$\Delta y = y_i - y_{i-1}$$

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$$

The stiffnesses of the above types of symmetrical arches are obtained by inverting the flexibility matrices.

The entire set of stiffnesses can be expressed in dimensionless form in combination with span L, rise r and rigidity EI of the segmental arch. Values of these coefficients were plotted in Figs. 16 through 20 for four types of symmetrical arches with rise to span ratios between 0.1 and 0.5 with the aid of a plotter subroutine. Since axial effect cannot be safely neglected at rise to span ratios up to 0.30, three values of I/AL^2 have been included in the investigation of this effect, they are 1/20000, 1/10000 and 1/5000. These values are considered to be representative. Stiffness values for other ratios of I/AL^2 may be obtained by interpolation.

The results obtained are in good agreement with others.[7,16,24,25]

2.6. Distribution Factors

Consider the general equations of static equilibrium for joint i of a structure, into which members ij, ik, ect. are rigidly connected as shown in Fig. 6. These are written as Eqs. (2.21).

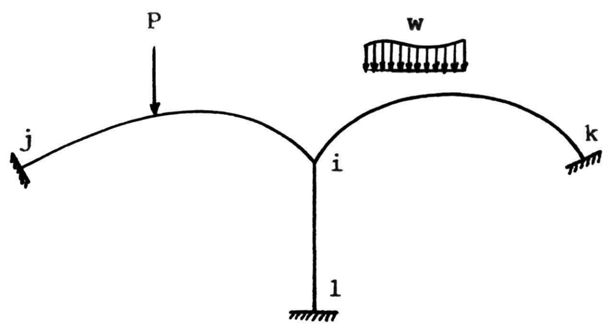


Fig. 6. Rigid Curved Frame

Now the forces and the moment acting on various member at a joint must be equal to any external forces and moment $\left\{F_i^E\right\}$ applied there thus: $\left\{F_i^E\right\} = \left\{\sum F_i\right\}$ represents this condition of static equilibrium. Substitution from Eqs. (2.21) gives

$$\left\{F_i^E\right\} = \left\{F_i^f\right\} + \left[\sum k_i\right]\left\{\delta_i\right\} + \sum\left(\left[r_i\right]\left\{\delta_j\right\}\right) \qquad (2.30)$$

If $\left\{F_i^E\right\} - \left\{F_i^f\right\}$ is written as $\left\{F_i^u\right\}$, it is clearly represents the equivalent external forces and moments acting at the joint i. With $\left\{F_i^E\right\} = 0$, the vector $\left\{F_i^u\right\}$ equals the balancing forces and moment which would have to be applied to the joint for equilibrium. Thus,

$$\left\{F_i^u\right\} = \left[\sum k_i\right]\left\{\delta_i\right\} + \sum\left(\left[r_i\right]\left\{\delta_j\right\}\right) \qquad (2.31)$$

If the joints are all first considered fixed throughout the structure, and joint i is released from restraint while all other joints remain fixed; then in Eq. (2.31) the displacement vectors $\left\{\delta_j\right\}$ are set at zero, and the first approximation to $\left\{\delta_i\right\}$ can be evaluated. The additional forces and moments occuring at the joint end of each of the members meeting at joint i, as a consequence of this release, can be evaluated by the generalized distribution matrix. That is,

$$\left\{F_i^d\right\} = \left[d_i\right]\left\{F_i^u\right\} \tag{2.32}$$

The distribution factor matrix is defined as follows:

$$\left[d_i\right] = -\left[k_i\right]\left[\sum k_i\right]^{-1} \tag{2.33.a}$$

For a symmetrical curved member which undergoes no vertical displacement at the springing ends, the distribution factors can be written in the following matrix form:

$$\left[d_i\right] = \begin{bmatrix} d11 & d13 \\ d31 & d33 \end{bmatrix} \tag{2.33.b}$$

in which

d11 = thrust distribution factor

d13 = thrust-moment distribution factor

d31 = moment-thrust distribution factor

d33 = generalized moment distribution factor

Eq. (2.33.a) can be expanded as follows:

$$\begin{bmatrix} d11 & d13 \\ d31 & d33 \end{bmatrix} = \begin{bmatrix} -k11 & -k13 \\ -k13 & -k33 \end{bmatrix}\begin{bmatrix} k11 & k13 \\ k31 & k33 \end{bmatrix}^{-1} \tag{2.33.c}$$

from which

$$\left.\begin{aligned} d11 &= -(k11\textstyle\sum k33 - k13\sum k31)/(\sum k11\sum k33 - \sum k13\sum k31) \\ d13 &= -(k13\textstyle\sum k11 - k11\sum k13)/(\sum k11\sum k33 - \sum k13\sum k31) \\ d31 &= -(k31\textstyle\sum k33 - k33\sum k31)/(\sum k11\sum k33 - \sum k13\sum k31) \\ d33 &= -(k33\textstyle\sum k11 - k31\sum k13)/(\sum k11\sum k33 - \sum k13\sum k31) \end{aligned}\right\} \tag{2.33.d}$$

It is at once apparent that the following relationships exist for the distribution factors at any particular joint:

$$\sum d11 = 1, \quad \sum d13 = 0, \quad \sum d31 = 0, \quad \sum d33 = 1 \qquad (2.33.e)$$

2.7 Carry-Over Factors

The effect of the release of joint i will accompany an effect on the adjacent joints. From Eq. (2.20) a system of forces equal to $[r_j]\{\delta_i\}$ will result at the end j of member ij. These induced forces are called carry-over forces and are related to the forces distributed to the joint end of the member by the following matrix expression:

$$[c_j] = [r_j][k_i]^{-1} \qquad (2.34.a)$$

For a symmetrical curved member for which the springings undergo no vertical displacement, the carry-over factor matrix can be written as follows:

$$[c_j] = \begin{bmatrix} c11 & c13 \\ c31 & c33 \end{bmatrix} \qquad (2.34.b)$$

in which

c11 = thrust carry-over factor

c13 = thrust-moment carry-over factor

c31 = moment-thrust carry-over factor

c33 = generalized moment carry-over factor

Values of various carry-over factors may be obtained by solving Eq. (2.34.a).

$$\begin{bmatrix} c11 & c13 \\ c31 & c33 \end{bmatrix} = \begin{bmatrix} r11 & r13 \\ r31 & r33 \end{bmatrix} \begin{bmatrix} k11 & k13 \\ k31 & k33 \end{bmatrix}^{-1} \qquad (2.34.c)$$

Substituting the relationship between restraint and stiffness factors from Eq. (2.18) gives

$$\begin{bmatrix} c11 & c13 \\ c31 & c33 \end{bmatrix} = \begin{bmatrix} -k11 & -k13 \\ -k31 & -k33-k23L \end{bmatrix} \begin{bmatrix} k11 & k13 \\ k31 & k33 \end{bmatrix}^{-1} \qquad (2.34.d)$$

which, when evaluated, results in the following elements:

$$\left.\begin{aligned} c11 &= -1 \\ c13 &= 0 \\ c31 &= k23*k31*L/(k11*k33 - k13*k31) \\ c33 &= -1 - k11*k23*L/(k11*k33 - k13*k31) \end{aligned}\right\} \qquad (2.34.e)$$

If the balance of a joint for moment is considered when only rotation can occur, the carry-over factor matrix reduces to one element matrix, that is, moment carry-over factor which has been mentioned in Section 2.3 by Eq. (2.17)

Moment carry-over factors may vary with several parameters, depending on the shape of the structural member, the restraint conditions and the axial effect considered. Values of moment carry-over factors of various types of symmetrical arches may be obtained from Fig. 21 in Appendix A.

2.8 Transmission Factors

When a system of forces and moments is applied at end A of a curved member AB, fractions of these forces and moments will transmit to end B. These quantities are called transmitted forces and moments. Thus, transmission factors are defined as the forces and the moments transmitted to the far end of a member when the near end is subjected to a system of unit forces and moments. In matrix notation, transmission factors can be written as

$$[t_i] = [c_i][d_i] \tag{2.35.a}$$

For a symmetrical curved member which has two independent joint movements (horizontal displacement and rotation), the transmission factor matrix is a 2 by 2 matrix

$$[t_i] = \begin{bmatrix} t11 & t13 \\ t31 & t33 \end{bmatrix} \tag{2.35.b}$$

in which

t11 = thrust transmission factor

t13 = thrust-moment transmission factor

t31 = moment-thrust transmission factor

t33 = generalized moment transmission factor

Elements of the transmission factor matrix can be obtained by expanding Eq. (2.35.a)

$$\left.\begin{aligned} t11 &= -d11 \\ t13 &= -d13 \\ t31 &= -d31 - \frac{k31\ k23*L}{\sum k11 \sum k33 - \sum k13 \sum k31} = -d31 - d21*L \\ t33 &= -d33 - \frac{k11\ k23*L}{\sum k11 \sum k33 - \sum k13 \sum k31} = -d33 - d23*L \end{aligned}\right\} \tag{2.35.c}$$

2.9 Thrust-Induction Factors

It is convenient to define a thrust-induction factor γ and a modified value γ' to allow for the adjustments in the horizontal forces at the joints during the moment-distribution procedure.

The thrust-induction factor for one end of a curved member is defined as the thrust developed in the member by a unit moment applied, without allowing translation, at one end when the other end is restrained against rotation.

$$\gamma_A = k13_A/k33_A \tag{2.36.a}$$

The modified thrust-induction factor is defined as the thrust due to a unit moment applied, without allowing translation, at one end of the curved member when the other end is free to rotate.

$$\gamma'_A = \gamma_A \frac{1 + c33_A}{1 - c33_A c33_B} \tag{2.36.b}$$

For the types of symmetrical member treated in this dissertation, the appropriate γ/r and γ'/r values are graphed in Figs. 22 and 23 in Appendix A.

2.10 Fixed-End Reactions

If initially all the joints of a structure are considered fixed in position and direction, the reactions (moments, thrusts and shear) required to act at the ends of a loaded member to maintain this fixity are called fixed-end reactions.

The algebraic sum of fixed-end moments or thrusts at any particular joint are not usually zero. These resultants of fixed-end moments or of fixed-end thrusts are called unbalanced moment or unbalanced thrust at that joint.

Like stiffness and carry-over factors, fixed-end reactions depend on the geometry, the structural properties of the curved member, and axial effect and restraint conditions.

The redundant reactions at the end A of a segmental arch may be obtained from the following matrix equation

$$\begin{Bmatrix} H_A \\ V_A \\ M_A \end{Bmatrix} = \begin{bmatrix} k11_A & k12_A & k13_A \\ k21_A & k22_A & k23_A \\ k31_A & k32_A & k33_A \end{bmatrix} \begin{Bmatrix} u^s_A \\ v^s_A \\ \theta^s_A \end{Bmatrix} \tag{2.37}$$

in which u^s_A, v^s_A and θ^s_A may be computed from Eqs. (2.5)

For a uniformly distributed load of w per unit length, values of moment and thrust due to external loads on the determinate base structure are evaluated as follows:

$$M^s = -wx^2/2 \tag{2.38.a}$$

$$H^s = -wx(dy/ds) \tag{2.38.b}$$

For a concentrated load P located at a point defined by X

$$M^s = -P(x - X), \quad \text{for } x > X \tag{2.39.a}$$

$$H^s = -P(dy/ds) \tag{2.39.b}$$

2.10.1 Parabolic Arch with Secant Variation in I

For a uniformly distributed load of w per unit length,

$$\begin{aligned} u^s_A &= -\frac{wrL^3}{10EI_o} \\ v^s_A &= \frac{wL^4}{8EI_o} \\ \theta^s_A &= \frac{wL^3}{6EI_o} \end{aligned} \tag{2.40}$$

For a concentrated load P with the location of the load defined as X

$$u_A^s = \frac{PrL^2}{3EI_o}\left(1 - \frac{X^4}{L^4} + \frac{2X^3}{L^3} - \frac{2X}{L}\right)$$

$$v_A^s = -\frac{PL^3}{6EI_o}\left(\frac{X^3}{L^3} - \frac{3X}{L} + 2\right) \qquad (2.41)$$

$$\theta_A^s = \frac{PL^2}{2EI_o}\left(-\frac{X^2}{L^2} + \frac{2X}{L} - 1\right)$$

2.10.2 Uniform Circular Arch

For a uniformly distributed load of w per unit length,

$$u_A^s = \frac{wR^4}{2EI}\left(-\frac{8}{3}\sin^3\beta - \frac{1}{2}\cos\beta\sin2\beta + \beta\cos\beta + 2\beta\cos\beta\sin^2\beta\right) + \frac{2wR^2}{3EA}\sin^3\beta$$

$$v_A^s = -\frac{wR^4}{2EI}\left(-2\beta\sin^3\beta + \frac{3}{2}\sin\beta\sin2\beta - 3\beta\sin\beta\right) - \frac{wR^2}{EA}\left(\frac{1}{2}\sin\beta\sin2\beta - \beta\sin\beta\right) \qquad (2.42)$$

$$\theta_A^s = -\frac{wR^3}{2EI}\left(-2\beta\sin^2\beta + \frac{1}{2}\sin2\beta - \beta\right)$$

For a concentrated load P located at a point defined by ω and X,

$$\begin{aligned} u_A^s = \frac{PR^2}{EI}\Big(& -R\sin\beta(\sin\beta + \sin\omega) - \frac{R}{2}(\sin^2\beta - \sin^2\omega) \\ & + X(\sin\beta + \sin\omega) + R\sin\beta\cos\beta(\beta + \omega) \\ & - R\cos\beta(\cos\beta - \cos\omega) + X\cos\beta(\beta + \omega)\Big) \\ & + \frac{PL}{2EA}\left(\sin^2\omega - \sin^2\beta\right) \end{aligned}$$

$$(2.43)$$

$$\begin{aligned} v_A^s = -\frac{PL^2}{EI}\Big(& -R\sin^2\beta(\beta + \omega) + 2R\sin\beta(\cos\beta - \cos\omega) \\ & + X\sin\beta(\beta + \omega) + \frac{R}{4}(\sin2\beta + \sin2\omega) \\ & - 0.5(\beta + \omega) - X(\cos\beta - \cos\omega)\Big) \end{aligned}$$

$$- \frac{PR}{EA}\left(\frac{1}{4}(\sin 2\beta + \sin 2\omega) - \frac{1}{2}(\beta + \omega)\right)$$

$$\theta_A^s = \frac{PR}{EI}\left(-R\sin\beta(\beta + \omega) + R(\cos\beta - \cos\omega) + X(\beta + \omega)\right)$$

2.10.3 Uniform Parabolic Arch

Applying Simpson's rule for the numerical integration to Eqs. (2.5), together with Eqs. (2.38) and (2.39), the displacements u_A^s, v_A^s and θ_A^s due to external loads on the statically determinate base structure can be computed.

2.10.4 Uniform Semi-Elliptical Arch

Using a finite element technique with the aid of an electronic digital computer, the displacements u_A^s, v_A^s and θ_A^s can be computed by the following finite summation expressions:

For a uniformly distributed load of w per unit length,

$$\begin{aligned} u_A^s &= \sum \frac{1}{2} wx^2 y \frac{\Delta s}{EI} + \sum -wx \frac{\Delta x}{EA} \frac{\Delta y}{\Delta s} \\ v_A^s &= \sum -\frac{1}{2} wx^3 \frac{\Delta s}{EI} + \sum -wx \frac{\Delta y}{EA} \frac{\Delta y}{\Delta s} \qquad (2.44) \\ \theta_A^s &= \sum -\frac{1}{2} wx^2 \frac{\Delta s}{EI} \end{aligned}$$

For a concentrated load P located at a point defined by X

$$\begin{aligned} u_A^s &= \sum P(x - X)y \frac{\Delta s}{EI} + \sum -P \frac{\Delta y}{EA} \frac{\Delta x}{\Delta s} \\ v_A^s &= \sum -P(x - X)x \frac{\Delta s}{EI} + \sum -P \frac{\Delta y}{EA} \frac{\Delta y}{\Delta s} \qquad (2.45) \\ \theta_A^s &= \sum -P(x - X) \frac{\Delta s}{EI} \end{aligned}$$

If the above deflection expressions are evaluated for a given case and these values substituted into Eq. (2.37), the corresponding values of H_A, V_A and M_A are obtained. Reactions at the end B of the segmental arch can be found by static equilibrium conditions.

Figs. 24 and 25 in Appendix A give coefficients for the fixed-end moment and the fixed-end thrust respectively for various types of symmetrical arches, with rise to span ratios between 0.1 and 0.5, when subjected to uniformly distributed load of w per unit length. Values of I/AL^2 used were 1/20000, 1/10000 and 1/5000.

Figs. 26 and 27 represent the influence lines for the fixed-end moment, fixed-end thrust and fixed-end shear for symmetrical arches (parabolic with secant variation in I, uniform circular, uniform parabolic and uniform semi-elliptical), all having a rise to span ratio at 0.25 and $I/AL^2 = 0$. Further information on influence line coefficients is given in Tables VI through XV. These tables contain the influence line coefficients for the fixed-end moment, fixed-end thrust and fixed-end shear for the same types of segmental arches with rise to span ratios from 0.1 to 0.5, in increments of 0.05, with $I/AL^2 = 0$. For uniform configurations, three values of I/AL^2 of 1/20000, 1/10000 and 1/5000 have been used to show the axial effect for rise to span ratios between 0.1 and 0.3.

The results obtained are in good agreement with previous publications.[16,26,27]

2.11 Modified Values of Stiffnesses and Fixed-End Moments

Modified values of stiffnesses, restraints and fixed-end reactions of a curved member are particular solutions of the general expressions. Such special conditions are often encountered in continuous systems where one or both end supports are free to rotate. The use of modified expressions usually leads to a reduced number of equations to be solved simultaneously or by numerical iterative procedures such as moment-distribution.

Consider a curve member AB where the support point A is fixed and the far end support B is hinged as shown in Fig. 7.

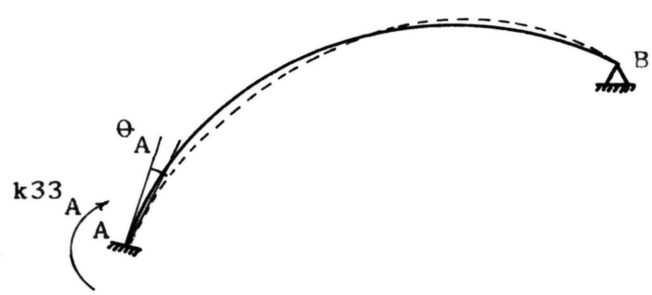


Fig. 7. Segmental Arch with a Hinged End

Suppose, however, that stiffness, restraint and carry-over factors and fixed-end moments are known for the fixed-end condition, and need to be deduced to allow for the presence of the hinge.

Assume that a unit rotation is imposed at the end A with the end B fixed, then there is a moment $k33_A$ developed at the end A and a moment $c33_A k33_A$ induced at the end B. The end B is then released to take up its position as a hinge, with the unit rotation at end A being maintained unchanged when the release at end B occurs. Then the moment distributed at B is $-c33_A k33_A$ and the moment carried over at end A is $-c33_B c33_A k33_A$. The modified moment stiffness factor at the end A is

$$k33'_A = k33_A(1 - c33_B c33_A) \tag{2.46}$$

which equals $k33_A(1 - c33^2_A)$ when the member is symmetrical.

Similarly, the modified fixed-end moment can be obtained by releasing the end B while maintaining the end A fixed. The fixed-end moments for both ends fixed are M^f_A and M^f_B; when end B is released with end A remaining fixed, the moment distributed at B is $-M^f_B$ and the moment carried over to A is $-c33_B M^f_B$. Thus, the modified fixed-end moment at the end A is

$$M^{f\prime}_A = M^f_A - c33_B M^f_B \tag{2.47}$$

The modified stiffness and restraint values may be obtained by a general approach, using the deformation-force matrix equation. If a curved member AB is fixed at the end A and hinged at the end B, obviously from Eq. (2.19), $M_B = 0$; then the moment stiffness at A may be determined by allowing the rotation θ_A to take place, where all other displacements are held to zero. The rotation at B exists since point B is hinged. Therefore,

$$M_B = r33_B \theta_A + k33_B \theta_B = 0$$

$$\theta_B = -r33_B \theta_A / k33_B$$

Then

$$M_A = k33_A \theta_A + r33_A \theta_B$$

$$= k33_A \theta_A + r33_A(-r33_B \theta_A)/k33_B$$

From this, the moment stiffness at A is (when $\theta_A = 1$)

$$k33'_A = k33_A - r33_A r33_B / k33_A \tag{2.48}$$

The above procedure illustrates that modified stiffness values may be obtained for any individual displacement; and solving for the corresponding values of θ_B and substituting in the proper relationship, the particular modified stiffness value may be determined. Then the modified deformation-force matrix equation may be shown as

$$\begin{Bmatrix} H_A \\ V_A \\ M_A \\ \cdots \\ H_B \\ V_B \end{Bmatrix} = \left[\begin{array}{ccc:cc} k11'_A & k12'_A & k13'_A & r11'_A & r12'_A \\ k21'_A & k22'_A & k23'_A & r21'_A & r22'_A \\ k31'_A & k32'_A & k33'_A & r31'_A & r32'_A \\ \hdashline r11'_B & r12'_B & r13'_B & k11'_B & k12'_B \\ r21'_B & r22'_B & r23'_B & k21'_B & k22'_B \end{array}\right] \begin{Bmatrix} u_A \\ v_A \\ \theta_A \\ \cdots \\ u_B \\ v_B \end{Bmatrix} \tag{2.49}$$

The modified fixed-end moment and fixed-end thrust values can be obtained by solving Eq. (2.49).

Chapter 3

THEORY OF INFINITE MATRIX SERIES IN CONTINUOUS CURVILINEAR STRUCTURES

3.1 Principle of Moment-Distribution

In moment-distribution method, the structure is first restrained by the fixed-end moments so that the ends of all members are held against rotation and translation. The unbalanced moments existing at the joints can be computed. These joints are then released one by one in such a manner that the unbalanced moment at each joint is distributed among the near ends and carried over to the far ends of the meeting members. These carry-over moments produce new unbalanced moments there. If the distribution and carry-over process is repeated an infinite number of times, the structure tends to its equilibrium state and the residual unbalanced moments tend to zero. Thus, moment-distribution is not an approximate method, but is a method of successive approximations by which exact results can be approached with any desired degree of accuracy. In other words, results obtained by the moment-distribution method can be expressed with mathematical rigor.

The final moment at a particular joint of a member is obtained by taking the sum of the initial fixed-end moment, the distributed moment and the carry-over moment at that end of the member. Thus,

$$\begin{aligned} \text{Final Moment} \quad = \quad & \text{Fixed-End Moment} \\ + \; & \text{Distributed Moment} \\ + \; & \text{Carry-Over Moment} \end{aligned}$$

Fixed-end moments are readily obtained by either force method or displacement method . Distributed and carry-over moments can be computed from unbalanced moments, which may be arranged in an infinite matrix series as will be developed in the subsequent sections of this dissertation.

3.2 Infinite Series Form of Carry-Over Moments

Consider a system of continuous curvilinear structure with two interior joints 1 and 2 as shown in Fig. 8. Since the ends of the arches and the column bases are assumed to be fixed, the rotation of joints 0, 3, 1′ and 2′ are known to be zero. Thus, only the two interior joints 1 and 2 are to be considered in the computations. As a secondary part of the solution, the moments at these fixed supports can be computed by a proper carry-over calculation according to the process of moment-distribution. If the end of the arches or the bases of the piers of the system were simply supported rather than fixed, the stiffness and the fixed-end moment of these members would be replaced by modified stiffness and fixed-end moment values as mentioned in Section 2.11. For the purpose of this dissertation, the joints are numbered by beginning with the first interior joint and continuing consecutively from left to right.

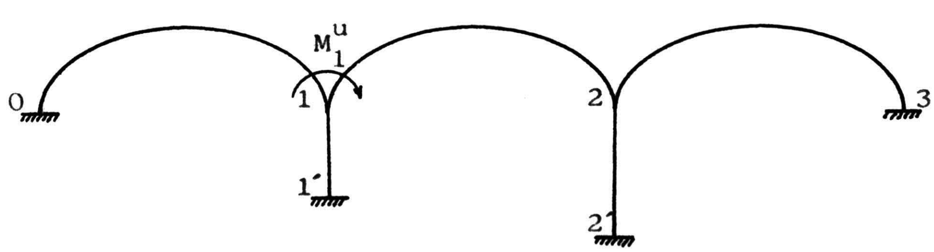


Fig. 8. Two-Span Continuous Arch Frame

Assuming that there exists an unbalanced moment $M_1^u = \sum M_1^f$ at joint 1, let d_{12}, c_{12} and d_{21}, c_{21} be the distribution and carry-over factors at the joints 1 and 2 of the member 12.

At the end of the first release, the moment relaxed at joint 2 is $-c_{12}d_{12}M_1^u = -t_{12}M_1^u$. If this unbalanced moment is in turn distributed and carried over for a second release, the moment relaxed at joint 1 is $c_{21}d_{21}c_{12}d_{12}M_1^u = \psi_{12}M_1^u$. The balancing operation completes its first cycle. In the same manner, if a second balancing cycle were made, the moments relaxed at the end of this cycle would be $\psi_{12}^2M_1^u$ and $-t_{12}\psi_{12}M_1^u$ at joints 1 and 2 respectively. It follows that if the relaxation procedure were to be continued for p cycles, the residual unbalanced moments at the end of the p-th cycle would be $\psi_{12}^pM_1^u$ and $-t_{12}\psi_{12}^{(p-1)}M_1^u$ at joints 1 and 2 respectively. When $p \to \infty$, these residual moments approach zero and the structural system takes its equilibrium state. The balancing operation is summarized in Table I.

The sums of all the moments relaxed at joints 1 and 2 can be written in the following infinite series form:

$$M_{12}^c = \left(\psi_{12} + \psi_{12}^2 + \psi_{12}^3 + \cdots + \psi_{12}^p\right) M_1^u \tag{3.1.a}$$

$$M_{21}^c = -t_{12}\left(1 + \psi_{12} + \psi_{12}^2 + \cdots + \psi_{12}^{p-1}\right) M_1^u \tag{3.1.b}$$

in which

t_{12} = moment transmission factor of member 12

ψ_{12} = flexural parameter of member 12

Since absolute values of distribution and carry-over factors of a member are always less than unity, then $-1 < \psi_{12} < 1$. The geometric

Table I. Balancing Operation

Joint	1	2
Member	1 – 2	
Distribution factor	d_{12}	d_{21}
Carry-over factor	c_{12}	c_{21}
Unbalanced moment	$M_1^u = \sum M_1^f$ at joint 1	
1^{st} cycle: 1^{st} release	$-d_{12}M_1^u$	$-c_{12}d_{12}M_1^u = -t_{12}M_1^u$
1^{st} cycle: 2^{nd} release	$\psi_{12}M_1^u = c_{21}d_{21}t_{12}M_1^u$	$d_{21}t_{12}M_1^u$
2^{nd} cycle: 3^{rd} release	$-d_{12}\psi_{12}M_1^u$	$-c_{12}d_{12}\psi_{12}M_1^u = -t_{12}\psi_{12}M_1^u$
2^{nd} cycle: 4^{th} release	$\psi_{12}^2M_1^u = c_{21}d_{21}t_{12}\psi_{12}M_1^u$	$d_{21}t_{12}\psi_{12}M_1^u$
-------------	----------------------	----------------------
p^{th} cycle: k-1 release	$-d_{12}\psi_{12}^{(k-1)}M_1^u$	$-t_{12}\psi_{12}^{(p-1)}M_1^u$
p^{th} cycle: k^{th} release	$\psi_{12}^pM_1^u$	$d_{21}t_{12}\psi_{12}^{(p-1)}M_1^u$
Sum of relaxed moment	$\dfrac{\psi_{12}}{1-\psi_{12}}M_1^u$	$-\dfrac{t_{12}}{1-\psi_{12}}M_1^u$

series (3.1.a) and (3.1.b) are convergent and consequently the sums of these sequences are respectively determined by the summation formula of an infinite series as follows:

$$M^c_{12} = \frac{\psi_{12}}{1 - \psi_{12}} M^u_1 \tag{3.2.a}$$

$$M^c_{21} = \frac{-t_{12}}{1 - \psi_{12}} M^u_1 \tag{3.2.b}$$

Similarly, the sums of all moments relaxed at 1 and 2 after balancing the unbalanced moment M^u_2 placed at the joint 2 are evaluated as follows:

$$M^c_{12} = \frac{-t_{12}}{1 - \psi_{12}} M^u_2 \tag{3.3.a}$$

$$M^c_{21} = \frac{\psi_{12}}{1 - \psi_{12}} M^u_2 \tag{3.3.b}$$

Utilizing the principle of superposition the carry-over moments at 1 and 2 are obtained from Eqs. (3.2) and (3.3).

$$M^c_{12} = \frac{\psi_{12}}{1 - \psi_{12}} M^u_1 - \frac{t_{12}}{1 - \psi_{12}} M^u_2 \tag{3.4.a}$$

$$M^c_{12} = \frac{-t_{12}}{1 - \psi_{12}} M^u_1 + \frac{\psi_{12}}{1 - \psi_{12}} M^u_2 \tag{3.4.b}$$

3.3 Restrained Infinite Matrix Series Method

3.3.1 Distribution Matrices of a Continuous System

Distribution factors at various joints of a continuous arch frame can be written in three separate matrices. The distribution matrix for the right side of all n interior joints of a continuous system is a n by n diagonal matrix as follows:

$$[D_{Ri}] = \begin{bmatrix} d_{R1} & 0 & 0 & \cdot & 0 \\ 0 & d_{R2} & 0 & \cdot & 0 \\ 0 & 0 & d_{R3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & d_{Rn} \end{bmatrix} \qquad (3.5.a)$$

Similarly, distribution matrices for the left side and for the column side of all interior joints of the continuous system are

$$[D_{Li}] = \begin{bmatrix} d_{L1} & 0 & 0 & \cdot & 0 \\ 0 & d_{L2} & 0 & \cdot & 0 \\ 0 & 0 & d_{L3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & d_{Ln} \end{bmatrix} \qquad (3.5.b)$$

$$[D_{Ci}] = \begin{bmatrix} d_{C1} & 0 & 0 & \cdot & 0 \\ 0 & d_{C2} & 0 & \cdot & 0 \\ 0 & 0 & d_{C3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & d_{Cn} \end{bmatrix} \qquad (3.5.c)$$

Note that $d_{R1} = d_{12}$, $d_{R2} = d_{23}$ ... and $d_{L1} = d_{10}$, $d_{L2} = d_{21}$, $d_{L3} = d_{32}$ ...

3.3.2 Transmission Matrices of a Continuous System

Elements of the transmission matrix of a continuous system, t_{ij}, is defined as the product of corresponding values of carry-over and distribution factors,

$$t_{ij} = c_{ij} d_{ij} \tag{3.6}$$

Eq. (3.6) produces the transmission matrix for the right side of all interior joints of a continuous system in the following form:

$$[T_{Ri}] = \begin{bmatrix} 0 & t_{R1} & 0 & \cdot & 0 \\ 0 & 0 & t_{R2} & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix} \tag{3.7.a}$$

in which $t_{R1} = t_{12}$, $t_{R2} = t_{23}$, ...

Similarly, the transmission matrix for the left side of all interior joints of a continuous system is

$$[T_{Li}] = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 \\ t_{L2} & 0 & 0 & \cdot & 0 \\ 0 & t_{L3} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix} \tag{3.7.b}$$

in which $t_{L2} = t_{21}$, $t_{L3} = t_{32}$, ...

The transmission matrix of a continuous system is

$$[T_i] = \begin{bmatrix} 0 & t_{R1} & 0 & \cdot & 0 \\ t_{L2} & 0 & t_{R2} & \cdot & 0 \\ 0 & t_{L3} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 \end{bmatrix} \tag{3.7.c}$$

3.3.3 Initial Unbalanced Moments

Initial unbalanced moments due to external loads can be expressed in the following vector:

$$\left\{M_i^u\right\}_o = \left\{\begin{matrix} M_1^u \\ M_2^u \\ \vdots \\ M_n^u \end{matrix}\right\}_o = \left\{\begin{matrix} M_{L1}^f + M_{R1}^f + M_{C1}^f \\ M_{L2}^f + M_{R2}^f + M_{C2}^f \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ M_{Ln}^f \quad M_{Rn}^f \quad M_{Cn}^f \end{matrix}\right\} \tag{3.8}$$

Initial unbalanced moments due to unit displacement of interior joints are written in the following matrix:

$$\left[m_{ij}^u\right]_o = \begin{bmatrix} m_{11}^u & m_{12}^u & 0 & \cdot & 0 \\ m_{21}^u & m_{22}^u & m_{23}^u & \cdot & 0 \\ 0 & m_{32}^u & m_{33}^u & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & m_{nn}^u \end{bmatrix}_o \tag{3.9}$$

where

$$m_{ii}^u = k13_{Li} + k13_{Ri} + k_{Ci}, \quad i = 1, n$$

$$m_{i,i+1}^u = k13_{Li+1}, \quad i = 1, n-1$$

$$m_{i,i-1}^u = k13_{Ri-1}, \quad i = 2, n$$

Other elements of the matrix (3.9) are set at zero.

3.3.4 Unbalanced Moments by Restrained Infinite Matrix Series Process

The first stage in this process is the determination of the end bending moments at joints of a continuous system, obtained by the balancing of the fixed-end moments induced by external loads on the various members. No linear displacements are allowed at any of the joints where such movements are possible. From the initial fixed-end forces and these end moments, the restraining forces required to act at the joints to prevent lateral displacements can be calculated. Next, for each possible linear joint displacement, it is necessary to obtain end moments corresponding to that particular unit displacement alone but with no rotational restraints occuring. The moments induced by the unit displacement considered but with no joint rotations or other displacements allowed must be first calculated; then the rotational restraints are removed as moments are distributed and carried over without any further linear displacement allowed. For each stage, with end moments so obtained, there is a corresponding set of restraining forces required to prevent linear movements at all those joints which are free to move.

The balancing process at each stage is as follows:

The moments relaxed at the end of the first release are

$$\left\{M_i^u\right\}_1 = \left[T'\right]\left\{M_i^u\right\}_o \tag{3.10.a}$$

where $\left[T'\right]$ is the transpose of the transmission matrix $\left[T\right]$.

The moments relaxed at the end of the second release are

$$\left\{M_i^u\right\}_2 = \left[T'\right]\left\{M_i^u\right\}_1 = \left[T'\right]^2\left\{M_i^u\right\}_o \tag{3.10.b}$$

In the same manner, the moments relaxed at the end of the k-th release are written as

$$\left\{M_i^u\right\}_k = \left[T'\right]^{k-1}\left\{M_i^u\right\}_{k-1} = \left[T'\right]^k\left\{M_i^u\right\}_o \tag{3.10.c}$$

The sum of all moments relaxed at joints of the continuous system due to external loads can be written in an infinite matrix series as follows:

$$\left\{M_i^u\right\}_s = \left(\left[T'\right] + \left[T'\right]^2 + \left[T'\right]^3 + \cdots + \left[T'\right]^k\right)\left\{M_i^u\right\}_o \quad (3.11)$$

The infinite matrix series within the parentheses is convergent, as will be shown in a subsequent section. The summation formula of the geometric matrix series gives

$$\left\{M_i^u\right\}_s = \left[T'\right]\left[\left[I\right] - \left[T'\right]\right]^{-1}\left\{M_i^u\right\}_o \quad (3.12)$$

Similarly, the sum of all moments relaxed at joints due to horizontal unit displacements at joints can be expressed in the following matrix equation:

$$\left[m_{ij}^u\right]_s = \left[T'\right]\left[\left[I\right] - \left[T'\right]\right]^{-1}\left[m_{ij}^u\right]_o \quad (3.13)$$

Hence, the total unbalanced moments are

$$\left\{M_i^u\right\}_t = \left\{M_i^f\right\} + \left\{M_i^u\right\}_s \quad (3.14)$$

$$\left[m_{ij}^u\right]_t = \left[m_i^f\right] + \left[m_{ij}^u\right]_s \quad (3.15)$$

3.3.5 Rotational Moments and Thrusts Due to Rotation

Once the total unbalanced moments at joints are known, the rotational moments due to external loads and horizontal unit displacements at joints are obtained by distribution and carry-over rules, it follows that

$$\left.\begin{aligned} \{M^r_{Ri}\} &= \Big[[D_{Ri}] + [T'_{Li}]\Big]\{M^u_i\}_t \\ \{M^r_{Li}\} &= \Big[[D_{Li}] + [T'_{Ri}]\Big]\{M^u_i\}_t \\ \{M^r_{Ci}\} &= [D_{Ci}]\{M^u_i\}_t \end{aligned}\right\} \tag{3.16}$$

and,

$$\left.\begin{aligned} [m^r_{Ri}] &= \Big[[D_{Ri}] + [T'_{Li}]\Big][m^u_{ij}]_t \\ [m^r_{Li}] &= \Big[[D_{Li}] + [T'_{Ri}]\Big][m^u_{ij}]_t \\ [m^r_{Ci}] &= [D_{Ci}][m^u_{ij}]_t \end{aligned}\right\} \tag{3.17}$$

The thrusts due to rotation can be computed by multiplying the change in rotational moments in a curved span by the thrust-induction factor γ or its modified value γ' as defined by Eqs. (2.36).

$$\left.\begin{aligned} \{H^r_{Ri}\} &= \gamma_{Ri}\{M^r_{Ri}\} - \gamma_{Li+1}\{M^r_{Li+1}\} \\ \{H^r_{Li}\} &= \gamma_{Li}\{M^r_{Li}\} - \gamma_{Ri-1}\{M^r_{Ri-1}\} \end{aligned}\right\} \tag{3.18}$$

$$\left.\begin{aligned} [h^r_{Ri}] &= \gamma_{Ri}[m^r_{Ri}] - \gamma_{Li+1}[m^r_{Li+1}] \\ [h^r_{Li}] &= \gamma_{Li}[m_{Li}] - \gamma_{Ri-1}[m^r_{Ri-1}] \end{aligned}\right\} \tag{3.19}$$

And for the column ii' of height L_{Ci},

$$\{H^r_{Ci}\} = \frac{1}{L_{Ci}}\Big(\{M^r_{Ci}\} + \{M^r_{Ci'}\}\Big) \tag{3.20}$$

$$[h^r_{Ci}] = \frac{1}{L_{Ci}}\Big([m^r_{Ci}] + [m^r_{Ci'}]\Big) \tag{3.21}$$

3.3.6 End Moments and End Thrusts

End moments and end thrusts are obtained by taking the sum of the initial fixed-end moments and the corresponding rotational moments. Thus, the end moments at joints due to external loads are

$$\left.\begin{aligned} \{M^e_{Ri}\} &= \{M^f_{Ri}\} + \{M^r_{Ri}\} \\ \{M^e_{Li}\} &= \{M^f_{Li}\} + \{M^r_{Li}\} \\ \{M^e_{Ci}\} &= \{M^f_{Ci}\} + \{M^r_{Ci}\} \end{aligned}\right\} \tag{3.22}$$

and the end moments at joints due to unit displacements at joints are

$$\left.\begin{aligned} [m^e_{Ri}] &= [m^f_{Ri}] + [m^r_{Ri}] \\ [m^e_{Li}] &= [m^f_{Li}] + [m^r_{Li}] \\ [m^e_{Ci}] &= [m^f_{Ci}] + [m^r_{Ci}] \end{aligned}\right\} \tag{3.23}$$

Similarly, end thrusts are computed as follows:

$$\left.\begin{aligned} \{H^e_{Ri}\} &= \{H^f_{Ri}\} + \{H^r_{Ri}\} \\ \{H^e_{Li}\} &= \{H^f_{Li}\} + \{H^r_{Li}\} \\ \{H^e_{Ci}\} &= \{H^f_{Ci}\} + \{H^r_{Ci}\} \end{aligned}\right\} \tag{3.24}$$

$$\left.\begin{aligned} [h^e_{Ri}] &= [h^f_{Ri}] + [h^r_{Ri}] \\ [h^e_{Li}] &= [h^f_{Li}] + [h^r_{Li}] \\ [h^e_{Ci}] &= [h^f_{Ci}] + [h^r_{Ci}] \end{aligned}\right\} \tag{3.25}$$

3.3.7 Equilibrium Equations at Joints

From Eqs. (3.24) and Eqs. (3.25) the unbalanced thrusts due to external loads and horizontal unit displacements at joints can be written as follows:

$$\left\{H_i^u\right\}_o = \left\{H_{Li}^e\right\} + \left\{H_{Ri}\right\} + \left\{H_{Ci}^e\right\} \tag{3.26}$$

$$\left[h_i^u\right]_o = \left[h_{Li}^e\right] + \left[h_{Ri}^e\right] + \left[h_{Ci}^e\right] \tag{3.27}$$

The conditions of equilibrium at joints can be utilized to establish the actual magnitudes of these horizontal displacements. Thus, if $\left\{u_i\right\}$ is the actual displacement vector, the equilibrium equations at joints are written as

$$\left[h_i^u\right]_o\left\{u_i\right\} + \left\{H_i^u\right\}_o = 0 \tag{3.28}$$

These conditions thus correspond to a set of n simultaneous equations for n interior joints involved; and the solution of these equations yields the displacement vector $\left\{u_i\right\}$.

3.3.8 Final Moments and Final Thrusts

With these values of displacements, the required end moments and end thrusts due to horizontal displacements are then computed by back-substituting; and according to the principle of superposition of forces the final moments and the final thrusts are obtained as follows:

$$\left\{M_i\right\} = \left\{M_i^e\right\} + \left[m_i^e\right]_o\left\{u_i\right\} \tag{3.29}$$

$$\left\{H_i\right\} = \left\{H_i^e\right\} + \left[h_i^e\right]_o\left\{u_i\right\} \tag{3.30}$$

3.3.9 Carry-Over Moments for Three and Four Span Continuous Systems

In order to illustrate the infinite matrix series approach, expressions of carry-over moments of continuous systems of three and four span are derived to compare with the previous works.

3.3.9.1 Three Span Continuous System

For a three span continuous system, refer to Fig. 8, with n = 2, carry-over moments are computed from Eqs. (3.12).

$$[T] = \begin{bmatrix} 0 & t_{12} \\ t_{21} & 0 \end{bmatrix}$$

$$\left[[I] - [T']\right]^{-1} = \frac{1}{1 - t_{12}t_{21}} \begin{bmatrix} 1 & t_{21} \\ t_{12} & 1 \end{bmatrix}$$

$$[T'_{Ri}]\left[[I] - [T']\right]^{-1} = \frac{1}{1 - \psi_{12}} \begin{bmatrix} 0 & 0 \\ -t_{12} & \psi_{12} \end{bmatrix}$$

$$[T'_{Li}]\left[[I] - [T']\right]^{-1} = \frac{1}{1 - \psi_{12}} \begin{bmatrix} \psi_{12} & -t_{21} \\ 0 & 0 \end{bmatrix}$$

$$M^c_{R1} = \frac{\psi_{12}}{1 - \psi_{12}} M^u_1 - \frac{t_{21}}{1 - \psi_{12}} M^u_2 \qquad (3.31.a)$$

$$M^c_{L2} = \frac{-t_{12}}{1 - \psi_{12}} M^u_1 + \frac{\psi_{12}}{1 - \psi_{12}} M^u_2 \qquad (3.31.b)$$

3.3.9.2 Four Span Continuous System

For four span continuous system, n = 3, it results as follows:

$$[T] = \begin{bmatrix} 0 & t_{12} & 0 \\ t_{21} & 0 & t_{23} \\ 0 & t_{32} & 0 \end{bmatrix}$$

$$\Big[[I] - [T']\Big]^{-1} = \frac{1}{1 - \psi_{12} - \psi_{23}} \begin{bmatrix} 1 - \psi_{23} & -t_{21} & t_{21}t_{32} \\ -t_{12} & 1 & -t_{32} \\ t_{12}t_{23} & -t_{23} & -t_{23}(1-\psi_{12}) \end{bmatrix}$$

$$[T'_{Ri}]\Big[[I] - [T']\Big]^{-1} = \frac{1}{1 - \psi_{12} - \psi_{23}} \begin{bmatrix} 0 & 0 & 0 \\ -t_{12}(1-\psi_{23}) & \psi_{12} & \psi_{12}t_{32} \\ t_{12}t_{23} & -t_{23} & \psi_{23} \end{bmatrix}$$

$$[T'_{Li}]\Big[[I] - [T']\Big]^{-1} = \frac{1}{1 - \psi_{12} - \psi_{23}} \begin{bmatrix} \psi_{12} & -t_{21} & t_{21}t_{32} \\ -\psi_{23}t_{12} & \psi_{23} & -t_{32}(1-\psi_{12}) \\ 0 & 0 & 0 \end{bmatrix}$$

If M_1^u, M_2^u and M_3^u are unbalanced moments at joints 1, 2 and 3 respectively, the carry-over moments at these joints are

$$M_{R1}^c = \frac{1}{1 - \psi_{12} - \psi_{23}} \left(\psi_{12}M_1^u - t_{21}M_2^u + t_{21}t_{32}M_3^u\right)$$

$$M_{R2}^c = \frac{1}{1 - \psi_{12} - \psi_{23}} \left(-\psi_{23}t_{12}M_1^u + \psi_{23}M_2^u - t_{32}(1 - \psi_{12})M_3^u\right)$$

$$M_{L2}^c = \frac{1}{1 - \psi_{12} - \psi_{23}} \left(-t_{12}(1 - \psi_{23})M_1^u + \psi_{12}M_2^u + \psi_{12}t_{32}M_3^u\right) \tag{3.32}$$

$$M_{L3}^c = \frac{1}{1 - \psi_{12} - \psi_{23}} \left(t_{12}t_{23}M_1^u - t_{23}M_2^u + \psi_{23}M_3^u\right)$$

These expressions are exactly the same as derived by others.[21,22]

3.4 Generalized Infinite Matrix Series Method

3.4.1 Matrix Analysis of Moment-Distribution

Suppose a number of straight and curved members, ij, ik, ect., are rigidly connected at joint i and are fixed-ended at the other ends, as shown in Fig. 6. If the various stiffness factors are known and the fixed-end moments and horizontal thrusts have been calculated at the ends of the various members for a given loading, it is evident that there are unbalanced moments and thrusts occuring at that joint. It is based upon the physical conception of allowing the various joints in turn both to rotate and displace horizontally together, so that the unbalanced moments and thrusts are successively eliminated at these joints. The general equation of equilibrium for the joint i of the structure may be written as follows:

$$\{F_i^u\} = [\sum k_i]\{\delta_i\} + \sum\left([r_i]\{\delta_j\}\right) \tag{3.33}$$

where

$\{F_i^u\}$ = generalized unbalanced force vector at joint i

$[\sum k_i]$ = matrix of sum of stiffness factors of various members framing into the joint i

$[r_i]$ = restraint factor matrix at joint i of a member framing from an adjacent joint

$\{\delta_i\}$ = displacement vector at joint i

Similar equations may be written for the other joints of the structure.

If all joints are first considered fixed throughout the structure, the unbalanced moments and forces at the various joints due to the given loads are determined from the fixed-end reactions. Suppose now that joint i is released from moment and thrust while all other joints remain fixed. Then in Eq. (3.33), $\{\delta_j\}$ is set at zero, and the first approximation to $\{\delta_i\}$ can be evaluated from the corresponding equation

$$\{F_i^u\} = [\sum k_i]\{\delta_i\} \tag{3.34}$$

The various additional forces and moments thus occuring at the end of each of the members meeting at joint i, in consequence of this release which results in generalized forces distributed to the near ends, may be evaluated by using distribution factor matrix equation (2.33.a).

The release of joint i will cause an accompanying effect on the adjacent joints. It is apparent from Eq. (2.20) that a generalized force vector equal to $[r_j]\{\delta_i\}$ will result at the end j of the member ij. These induced generalized forces are said to be carried over and can be computed by using carry-over factor matrix equation (2.34.a).

This procedure of successively releasing joints, distributing the unbalanced generalized forces and carrying over related forces to the adjacent joints by using matrix operation to generalize the procedure, is described as the matrix analysis of moment-distribution.

3.4.2 Distribution Matrices

In the case of continuous curvilinear frames, there are two degrees of freedom considered at each joint, the rotation and the horizontal displacement of the joint. If the joint is released for both movements simultaneously, the distribution factors constitute a 2 by 2 matrix at each joint. Thus, the distribution matrices for the structure are similar to those defined by Eqs. (3.5.a), (3.5.b) and (3.5.c) except that each diagonal element is replaced by a 2 by 2 matrix which is defined as Eq. (2.33.c). For example, the distribution factor matrix for the right side of all interior joints of a continuous arch frame is written as follows:

$$[D_{Ri}] = \begin{bmatrix} d11_{R1} & d13_{R1} & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ d31_{R1} & d33_{R1} & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & d11_{R2} & d13_{R2} & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & d31_{R2} & d33_{R2} & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & d11_{R3} & d13_{R3} & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & d31_{R3} & d33_{R3} & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & d11_{Rn} & d13_{Rn} \\ 0 & 0 & 0 & 0 & 0 & 0 & & d31_{Rn} & d33_{Rn} \end{bmatrix} \qquad (3.35)$$

The distribution factor matrices for the left side and for the column side of all interior joints of a continuous arch frame are established in a similar manner.

3.4.3 Transmission Matrices

In the generalized case, joints are released one by one which allows both translation and rotation simultaneously, so that the transmission factor matrix for a continuous curvilinear structure of n interior joints is a 2n by 2n matrix. Making use of Eqs. (2.35.b), (2.35.c) and Eq. (3.7.c), the transpose of the transmission matrix of a continuous system is assembled as follows:

$$[T_i'] =$$

$$\left[\begin{array}{cccccccccccc}
0 & 0 & t11_{L2} & t13_{L2} & 0 & 0 & \vdots & 0 & 0 & 0 & 0 \\
0 & 0 & t31_{L2} & t33_{L2} & 0 & 0 & \vdots & 0 & 0 & 0 & 0 \\
t11_{R1} & t13_{R1} & 0 & 0 & t11_{L3} & t13_{L3} & \vdots & 0 & 0 & 0 & 0 \\
t31_{R1} & t33_{R1} & 0 & 0 & t31_{L3} & t33_{L3} & \vdots & 0 & 0 & 0 & 0 \\
0 & 0 & t11_{R2} & t13_{R2} & 0 & 0 & \vdots & 0 & 0 & 0 & 0 \\
0 & 0 & t31_{R2} & t33_{R2} & 0 & 0 & \vdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & t11_{Ln} & t13_{Ln} \\
0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & t31_{Ln} & t33_{Ln} \\
0 & 0 & 0 & 0 & 0 & 0 & \vdots & t11_{Rn-1} & t13_{Rn-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \vdots & t31_{Rn-1} & t33_{Rn-1} & 0 & 0
\end{array}\right]$$

(3.36)

The transposed transmission matrices for the right side and for the left side of a continuous system can be established in a similar manner.

3.4.4 Unbalanced Generalized Forces by Infinite Matrix Series

The initial unbalanced thrusts and moments at n joints of a continuous system can be expressed in a generalized force vector as

$$\left\{F_i^u\right\}_o = \left\{\begin{array}{c} H_1^u \\ M_1^u \\ \cdot\cdot \\ H_n^u \\ M_n^u \end{array}\right\}_o = \left\{\begin{array}{ccccc} H_{L1}^f & + & H_{R1}^f & + & H_{C1}^f \\ M_{L1}^f & + & M_{R1}^f & + & M_{C1}^f \\ \cdots & & \cdots & & \cdots \\ H_{Ln}^f & + & H_{Rn}^f & + & H_{Cn}^f \\ M_{Ln}^f & + & M_{Rn}^f & + & M_{Cn}^f \end{array}\right\} \qquad (3.37)$$

Thrusts and moments relaxed at the end of the first release in the matrix moment-distribution procedure are

$$\left\{F_i^u\right\}_1 = \left[T'\right]\left\{F_i^u\right\}_o \tag{3.38.a}$$

Thrusts and moments relaxed at the end of the k-th release are

$$\left\{F_i^u\right\}_2 = \left[T'\right]\left\{F_i^u\right\}_1 = \left[T'\right]^2\left\{F_i^u\right\}_o \tag{3.38.b}$$

Thrusts and moments relaxed at the end of the k-th release are deduced as follows:

$$\left\{F_i^u\right\}_k = \left[T'\right]^{k-1}\left\{F_i^u\right\}_{k-1} = \left[T'\right]^k\left\{F_i^u\right\}_o \tag{3.38.c}$$

The first cycle of balancing operation is complete after releasing n joints of the continuous system. If the balancing were to be continued for p cycles, the residual unbalanced thrusts and moments at the end of the p-th cycle would be

$$\left\{F_i^u\right\}_{np} = \left[T'\right]^{np}\left\{F_i^u\right\}_o \tag{3.38.d}$$

The sum of all generalized forces (moments and thrusts) relaxed at joints of the continuous system can be written in an infinite matrix series as follows:

$$\left\{F_i^u\right\}_s = \left(\left[T'\right] + \left[T'\right]^2 + \left[T'\right]^3 + \cdots + \left[T'\right]^k\right)\left\{F_i^u\right\}_o \tag{3.39}$$

The infinite matrix series within the parentheses is convergent, as will be shown in the following section. The summation formula of the geometric matrix series gives

$$\left\{F_i^u\right\}_s = [T'][[I]-[T']]^{-1}\left\{F_i^u\right\}_o \quad (3.40)$$

The total unbalanced generalized forces are the sums of the generalized fixed-end forces and the corresponding unbalanced generalized forces.

$$\left\{F_i^u\right\}_t = \left\{F_i^f\right\} + \left\{F_i^u\right\}_s \quad (3.41)$$

3.4.5 Final Moments and Final Thrusts

The final moments and thrusts at joints are obtained by the rule of distribution and carry-over. It follows that

$$\left\{F_{Ri}\right\} = \left\{F_{Ri}^f\right\} + \left([D_{Ri}] + [T'_{Li}]\right)\left\{F_i^u\right\}_t \quad (3.42.a)$$

$$\left\{F_{Li}\right\} = \left\{F_{Li}^f\right\} + \left([D_{Li}] + [T'_{Ri}]\right)\left\{F_i^u\right\}_t \quad (3.42.b)$$

$$\left\{F_{Ci}\right\} = \left\{F_{Ci}^f\right\} + [D_{Ci}]\left\{F_i^u\right\}_t \quad (3.42.c)$$

3.5 Proof of the Convergence of the Transmission Matrices

After the k-th release operation there remains at joint i of a continuous system a residual force vector $\left\{F_i^u\right\}_k = [T']^k\left\{F_i^u\right\}_o$. It will be demonstrated that these terms approach zero when k increases infinitely; that is to say, the infinite matrix series

$$[S_k] = [I] + [T'] + [T']^2 + [T']^3 + \cdots + [T']^k \quad (3.43)$$

is convergent.

R. Oldenburger[28] has proved by matrix method the convergence of successive moment-distribution for beams, C. Massonnet[29] has treated analytically the convergence for frames by using the Jacobi iterative method; and N.J. Hoff[30] has shown that in each cycle of the moment-distribution procedure, consisting of balancing, distribution, and carry-over operations, the total potential of the system decreases to a minimum value compatible with the requirement that the far ends of the members be rigidly fixed.

It will be shown here that the moment-distribution process is convergent in terms of the transmission matrix sequence. Some definitions and theorems connected with the matrix limit concept are demonstrated in the Appendix B for convenient use.

In order that $[T]^k \rightarrow 0$, it is sufficient that at least one of norms of the matrix $[T]$ be less than one. By using this condition, it is observed at first glance that the transmission matrix $[T]$ defined in Eq. (3.7.c) has its first norm less than one in modulus, since any element t_{ij} is less than 0.5 (carry-over factor is less than 0.5 and distribution factor is always less than unity). It is evident that the transmission matrix $[T]^k \rightarrow 0$ when $k \rightarrow \infty$; thus for the restrained infinite matrix series approach, the balancing process is convergent.

This condition is sufficient but by no means necessary. This can not apply to the generalized transmission matrix defined in Eq. (3.36). Another sharper criterion of convergence should be established. A necessary and sufficient condition that $[T]^k \rightarrow 0$ as k increases is that all the eigenvalues of $[T]$ be less than unity. The proof is included in the Appendix B. This criterion may be applied to the generalized transmission matrix (3.36). A subroutine is added to the computer program in order to test the convergence of the transmission matrix by finding its largest eigenvalue.

An alternate technique is that by using Sylvester 's theorem, the largest eigenvalue can be approximated by computing the ratio $t_{ij}^{(k)}/t_{ij}^{(k-1)}$ between corresponding elements in the matrices $[T']^k$ and $[T']^{k-1}$. The approximate value of the largest eigenvalue is then

$$\lambda = \left(\frac{t_{ij}^{(2k)}}{t_{ij}^{(k)}}\right)^{\frac{1}{k}} \tag{3.44}$$

3.6 Approximations. Errors Committed in Stopping at Any Stage

As it has been shown, infinite matrix series methods are exact methods which involve the computation of the inverse of a matrix. This is equivalent to the slope-deflection method. However, since successive powers of $[T']$ correspond to successive release of joints in the moment-distribution process, approximate results can be obtained by taking the partial sum of a finite number of terms of the infinite matrix series in $[T']$.

In this case, it is necessary to investigate the errors that may be committed in stopping at any stage of balancing and the rate of convergence of the process so that one may know how many steps must be taken to obtain a given accuracy.

Suppose that after k times of joint release, $[S_k]$ is the partial sum up to the k-th power of $[T']$ of the series (3.43), and that $[S_k]$ is a good enough approximation to the inverse of the matrix $[Q] = [I] - [T']$

Starting with this approximate sum $[S_k]$, an iterative procedure can be applied to generate a sequence of matrices $[S_{k+1}]$, $[S_{k+2}]$, ect.

$$[S_{k+1}] = [S_k] + [E_k] \tag{3.45}$$

where $[E_k]$ is the error matrix of $[S_k]$ at the end of the k-th cycle of release. The error matrix $[E_k]$ can be defined by the following relationship:

$$[E_k] = [I] - [Q][S_k] \tag{3.46}$$

Since $[S_k]$ is very close to $[Q]^{-1}$, it is assumed that all eigenvalues of $[E_k]$ are less than unity in absolute value; and it follows that

$$\| E_{k+1} \| \leqslant \| E_k \| \tag{3.47}$$

Thus, when k increases infinitely, it is evident from Eq. (3.46) that the sequence $[S_k]$ approaches $[Q]^{-1}$ and an exact solution is obtained.

From the recursion formulas (3.45) and (3.46), the error that may be committed in stopping at any stage of balancing can be evaluated as follows:

Suppose, at the end of the $(k+\nu)$-th cycle of release, elements of the error matrix are small enough to be neglected; the total error matrix is

$$[E_t] = [E_k] + [E_{k+1}] + \cdots + [E_{k+\nu}] \tag{3.48}$$

Thus, from the rule of distribution and carry-over of forces, the error vectors of the generalized forces committed at the end of the $(k+\nu)$-th cycle can be computed as follows:

$$\left\{e_{Ri}\right\} = \left(\left[D_{Ri}\right] + \left[T'_{Li}\right]\right)\left[T'\right]\left[E_t\right]\left\{F_i^u\right\}_o \tag{3.49.a}$$

$$\left\{e_{Li}\right\} = \left(\left[D_{Li}\right] + \left[T'_{Ri}\right]\right)\left[T'\right]\left[E_t\right]\left\{F_i^u\right\}_o \tag{3.49.b}$$

$$\left\{e_{Ci}\right\} = \left[D_{Ci}\right]\left[T'\right]\left[E_t\right]\left\{F_i^u\right\}_o \tag{3.49.c}$$

Illustrative examples are given in Chapter 4.

3.7 Estimate of Rate of Convergence of the Balancing Process

The rate of convergence of the balancing process is that of the infinite matrix series in $[T]$. The matrix norm concept is useful in estimating the rate of convergence of a matrix series.

For the case of the restrained infinite matrix series, the first norm of the transmission matrix is always less than unity; then the inequality (B.12) in Appendix B can be used to estimate the rate of convergence of the balancing process.

For the case of the generalized infinite matrix series, the norm of the transmission matrix is not usually less than unity. The estimate for convergence is based on the largest eigenvalue or the third norm of the error matrices.

Some numerical examples are given in Chapter 4 to illustrate these criteria.

Chapter 4

APPLICATION OF ANALYSIS

4.1 Scope

In this chapter, the information obtained from Chapter 2 and the theory formulated in Chapter 3 will be applied to the analysis of continuous curvilinear structures.

A numerical example is chosen to illustrate the application of the two proposed methods.

Approximations can be made by using partial sums of the infinite matrix series. Errors that may be committed in stopping at any stage are analyzed numerically and the rate of convergence of the methods is estimated.

Computer programs are developed by using infinite matrix series approaches to solve complex problems. Illustrative numerical examples are given.

Finally, an experimental model was built and tested in order to provide the necessary correlation between the analytical solution of the mathematical model and experimental results from the physical model.

4.2 Procedure

The application of the infinite matrix series methods to the analysis of plane continuous curvilinear structures consists of the following steps:

(1) Determine stiffness factors of segmental arches which

constitute the continuous system, from graphs 16 through 20 in the Appendix A. If the restrained method is to be used, carry-over and thrust-induction factors can be taken from graphs 21 through 23.

(2) Compute distribution and transmission factors of segmental arches by Eqs. (2.33.d) and Eqs. (2.35.c). Form the distribution and transmission matrices for the structure as in Eqs. (3.5) and (3.7) for the restrained method, or as in Eqs. (3.35) and (3.36) for the generalized method.

(3) Determine fixed-end moments and fixed-end thrusts in terms of external loads by using graphs 24 through 27 and tables VI through XV. Form the fixed-end moment and fixed-end thrust matrices.

(4) Compute the unbalanced moments and thrusts at joints based on the infinite matrix series summation expressions (3.12) through (3.15) for the restrained method, or expressions (3.40) and (3.41) for the generalized method.

(5) For the restrained method, (a) compute the rotational moments and the thrusts due to rotation, (b) write as many independent equilibrium equations as there are interior joints, and (c) solve simultaneously for unknown horizontal displacements.

(6) Compute the final moments and the final thrusts.

The sign convention adopted in this dissertation is the same as used in the slope-deflection method. The positive direction for the end moments, angles and rotational displacements is taken as clockwise. Forces, distances and linear displacements are taken as positive when measured from left to right or vertically upward.

4.3 Mathematical Model

4.3.1 Restrained Infinite Matrix Series Method

The restrained method will be applied to the continuous arch system shown in Fig. 9.

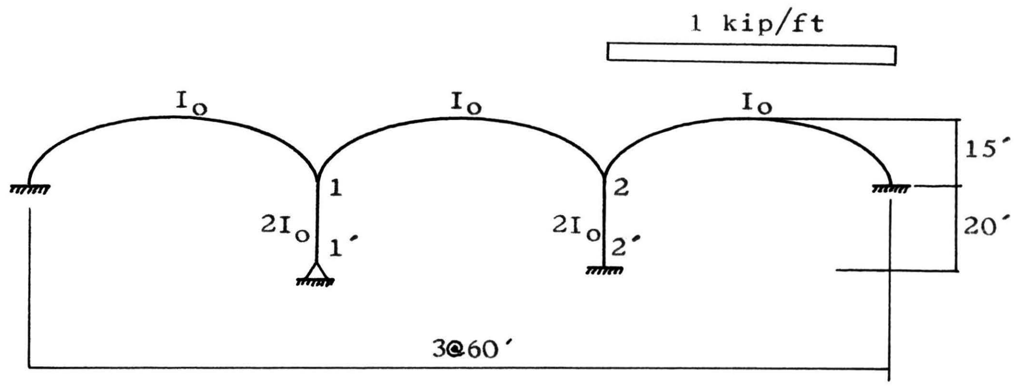


Fig. 9. Continuous Arch System

The system is composed of three similar symmetrical arches. Each arch member is a uniform semi-elliptical shape with a 60 ft span and 15 ft rise; and the moment of inertia of the cross-section is I_o. The left and right ends of the system are fixed and the interior joints 1 and 2 are rigidly connected with columns of 20 ft height. The base of the first column is hinged and the other is fixed. The moment of inertia of the cross-section of each column is $2I_o$. A uniform load of 1 kip per foot is applied over the right-hand arch. The analysis includes the axial effect, with the I/AL^2 ratio for each arch member taken as 1/10000.

Solution

(1) For the arch, moment stiffness, carry-over and thrust-

induction factors are taken from graphs 18, 21 and 23 with $r/L = 0.25$ and $I/AL^2 = 1/10000$. For convenience, let $EI_o = 100$.

$$k33_{R1} = k33_{L2} = 7.75 \frac{EI_o}{L} = 7.75 \frac{100}{60} = 12.9167$$

$$c33_{R1} = c33_{L2} = -0.4903$$

$$\gamma'_{R1} = \gamma'_{L2} = \frac{0.607}{r} = \frac{0.607}{15} = 0.0405$$

For the column 11′, the modified moment stiffness value is

$$k33'_{C1} = \frac{3EI_c}{L_{C1}} = \frac{3*200}{20} = 30$$

For the column 22′,

$$k33_{C2} = \frac{4EI_c}{L_{C2}} = \frac{4*200}{20} = 40$$

(2) Distribution and transmission factors are

$$d33_{L1} = d33_{R1} = \frac{-k33_{L1}}{\sum k33_1} = -\frac{12.9167}{55.8333} = -0.2313$$

$$d33_{C1} = \frac{-k33_{C1}}{\sum k33_1} = -\frac{30}{55.8333} = -0.5373$$

$$d33_{L2} = d33_{R2} = \frac{-k33_{L2}}{\sum k33_2} = -\frac{12.9167}{65.8333} = -0.1962$$

$$d33_{C2} = \frac{-k33_{C2}}{\sum k33_2} = -\frac{40}{65.8333} = -0.6076$$

$$t33_{R1} = c33_{R1}d33_{R1} = (-.4903)(-.2313) = 0.1134$$

$$t33_{L2} = c33_{L2}d33_{L2} = (-.4903)(-.1962) = 0.0962$$

From the above computed factors, the distribution and transmission factor matrices for the structure are formed as follows:

$$[D_{Ri}] = [D_{Li}] = \begin{bmatrix} -.2313 & 0 \\ 0 & -.1962 \end{bmatrix}$$

$$[D_{Ci}] = \begin{bmatrix} -.5373 & 0 \\ 0 & -.6076 \end{bmatrix}$$

$$[T_{Ri}] = \begin{bmatrix} 0 & 0.1134 \\ 0 & 0 \end{bmatrix}$$

$$[T_{Li}] = \begin{bmatrix} 0 & 0 \\ 0.0962 & 0 \end{bmatrix}$$

$$[T_i] = \begin{bmatrix} 0 & 0.1134 \\ 0.0962 & 0 \end{bmatrix}$$

(3) Fixed-end moments and fixed-end thrusts due to external loads can be taken from graphs 24 and 25.

$$M^f_{R1} = M^f_{L1} = M^f_{L2} = M^f_{C1} = M^f_{C2} = 0$$

$$M^f_{R2} = 0.0289\ wL^2 = 0.0289*60*60 = 104.04 \text{ ft-kips}$$

$$H^f_{R1} = H^f_{L1} = H^f_{L2} = H^f_{C1} = H^f_{C2} = 0$$

$$H^f_{R2} = 0.1438\ \frac{wL^2}{r} = 0.1438*60*60/15 = 34.512 \text{ kips}$$

Fixed-end moments and fixed-end thrusts due to horizontal unit displacement at joints are taken from graphs 19 and 16, respectively.

Let $u_1 = 1$

$$m^f_{R1} = m^f_{L1} = -m^f_{L2} = 7.02\ \frac{EI_o}{Lr} = 7.02\ \frac{100}{60*15} = 0.7800 \text{ ft-kips}$$

$$m^f_{C1} = \frac{-3EI_c}{L^2_{C1}} = -\ \frac{3*200}{20*20} = -1.5 \text{ ft-kips}$$

$$m^f_{R2} = m^f_{C2} = 0$$

$$h^f_{R1} = h^f_{L1} = -h^f_{L2} = 9.95\ \frac{EI_o}{Lr^2} = 9.95\ \frac{100}{60*15*15} = 0.0735 \text{ kips}$$

$$h^f_{C1} = \frac{3EI_c}{L^3_{C1}} = \frac{200}{20*20*20} = 0.0750 \text{ kips}$$

$$h^f_{R2} = h^f_{C2} = 0$$

Let $u_2 = 1$

$$m^f_{R2} = m^f_{L2} = -m^f_{R1} = 0.7800 \text{ ft-kips}$$

$$m^f_{C2} = \frac{-6EI_c}{L^2_{C2}} = -\ \frac{6*200}{20*20} = -3.0 \text{ ft-kips}$$

$$m^f_{L1} = m^f_{C1} = 0$$

$$h^f_{R2} = h^f_{L2} = -h^f_{R1} = 0.0735 \text{ kips}$$

$$h^f_{C2} = \frac{12EI_c}{L^3_{C2}} = \frac{12*200}{20*20*20} = 0.3000 \text{ kips}$$

$$h^f_{L1} = h^f_{C1} = 0$$

The fixed-end moments and fixed-end thrusts due to external loads may be expressed in the following vectors

$$\left\{M^f_{Ri}\right\} = \begin{Bmatrix} 0 \\ 104.04 \end{Bmatrix}, \quad \left\{M^f_{Li}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \left\{M^f_{Ci}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left\{H^f_{Ri}\right\} = \begin{Bmatrix} 0 \\ 34.512 \end{Bmatrix}, \quad \left\{H^f_{Li}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \left\{H^f_{Ci}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Also, the fixed-end moments and the fixed-end thrusts due to unit displacement at joints are expressed in the following matrices:

$$\left[m^f_{Ri}\right] = \begin{bmatrix} 0.78 & -0.78 \\ 0 & 0.78 \end{bmatrix}, \quad \left[h^f_{Ri}\right] = \begin{bmatrix} 0.0735 & -0.0735 \\ 0 & 0.0735 \end{bmatrix}$$

$$\left[m^f_{Li}\right] = \begin{bmatrix} 0.78 & 0. \\ -0.78 & 0.78 \end{bmatrix}, \quad \left[h^f_{Li}\right] = \begin{bmatrix} 0.0735 & 0. \\ -0.0735 & 0.0735 \end{bmatrix}$$

$$\left[m^f_{Ci}\right] = \begin{bmatrix} -1.50 & 0. \\ 0. & -3.00 \end{bmatrix}, \quad \left[h^f_{Ci}\right] = \begin{bmatrix} -0.075 & 0. \\ 0. & -0.3000 \end{bmatrix}$$

(4) Computation of unbalanced moments

The initial unbalanced moment vector due to external loads is

$$\left\{M^u_i\right\}_o = \begin{Bmatrix} 0. \\ 104.04 \end{Bmatrix}$$

and the initial unbalanced moment matrix due to unit displacements is

$$\left[m^u_{ij}\right]_o = \begin{bmatrix} 0.06 & -0.78 \\ -0.78 & -1.44 \end{bmatrix}$$

The sums of moments relaxed at joints due to external loads and due to unit displacements are given by the infinite matrix series summation formulas (3.12) and (3.13). Substitution gives

$$\left\{M_i^u\right\}_s = \begin{bmatrix} 0 & .0962 \\ .1134 & 0 \end{bmatrix} \begin{bmatrix} 1 & -.0962 \\ -.1134 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 0. \\ 104.04 \end{Bmatrix} = \begin{Bmatrix} 10.1193 \\ 1.1479 \end{Bmatrix}$$

$$\left[m_{ij}^u\right]_s = \begin{bmatrix} 0 & .0962 \\ .1134 & 0 \end{bmatrix} \begin{bmatrix} 1 & -.0962 \\ -.1134 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.06 & -0.78 \\ -.78 & -1.44 \end{bmatrix}$$

$$= \begin{bmatrix} -.0752 & -.1487 \\ -.0017 & -.1053 \end{bmatrix}$$

The total unbalanced moments at joints are given by Eqs. (3.14) and (3.15)

$$\left\{M_i^u\right\}_t = \begin{Bmatrix} 0. \\ 104.04 \end{Bmatrix} + \begin{Bmatrix} 10.1193 \\ 1.1479 \end{Bmatrix} = \begin{Bmatrix} 10.1193 \\ 105.1878 \end{Bmatrix}$$

$$\left[m_i^u\right]_t = \begin{bmatrix} 0.06 & -0.78 \\ -.78 & -1.44 \end{bmatrix} + \begin{bmatrix} -.0752 & -.1487 \\ -.0017 & -.1053 \end{bmatrix} = \begin{bmatrix} -.0152 & -.9287 \\ -.7817 & -1.5453 \end{bmatrix}$$

(5a) Rotational moments as computed from Eqs. (3.16) and (3.17) result in the following matrices:

$$\left\{M_{Ri}^r\right\} = \begin{bmatrix} -.2313 & 0.0962 \\ 0 & -.1962 \end{bmatrix} \begin{Bmatrix} 10.1193 \\ 105.1878 \end{Bmatrix} = \begin{Bmatrix} 7.7783 \\ -20.6381 \end{Bmatrix}$$

$$\left\{M_{Li}^r\right\} = \begin{bmatrix} -.2313 & 0 \\ 0.1134 & -.1962 \end{bmatrix} \begin{Bmatrix} 10.1193 \\ 105.1878 \end{Bmatrix} = \begin{Bmatrix} -2.3410 \\ -19.4902 \end{Bmatrix}$$

$$\left\{M^r_{Ci}\right\} = \begin{bmatrix} -.5373 & 0 \\ 0 & -.6076 \end{bmatrix} \begin{Bmatrix} 10.1193 \\ 105.1878 \end{Bmatrix} = \begin{Bmatrix} -5.4373 \\ -63.9116 \end{Bmatrix}$$

$$\left[m^r_{Ri}\right] = \begin{bmatrix} -.2313 & .0962 \\ 0 & -.1962 \end{bmatrix} \begin{bmatrix} -.0152 & -.9287 \\ -.7817 & -1.5453 \end{bmatrix} = \begin{bmatrix} -.0717 & .0662 \\ .1534 & .3032 \end{bmatrix}$$

$$\left[m^r_{Li}\right] = \begin{bmatrix} -.2313 & 0 \\ 0.1134 & -.1962 \end{bmatrix} \begin{bmatrix} -.0152 & -.9287 \\ -.7817 & -1.5453 \end{bmatrix} = \begin{bmatrix} 0.0035 & .2148 \\ 0.1517 & .1979 \end{bmatrix}$$

$$\left[m^r_{Ci}\right] = \begin{bmatrix} -.5373 & 0 \\ 0 & -.6076 \end{bmatrix} \begin{bmatrix} -.0152 & -.9287 \\ -.7817 & -1.5453 \end{bmatrix} = \begin{bmatrix} 0.0082 & .4990 \\ 0.4750 & .9389 \end{bmatrix}$$

Similarly, thrusts due to rotation are calculated from Eqs. (3.18) through (3.21).

$$\left\{H^r_{Ri}\right\} = 0.0405 \left(\begin{Bmatrix} 7.7783 \\ -20.6381 \end{Bmatrix} - \begin{Bmatrix} -19.4902 \\ 10.0539 \end{Bmatrix} \right) = \begin{Bmatrix} 1.1049 \\ -1.2463 \end{Bmatrix}$$

$$\left\{H^r_{Li}\right\} = 0.0405 \left(\begin{Bmatrix} -2.3410 \\ -19.4902 \end{Bmatrix} - \begin{Bmatrix} 1.1478 \\ 7.7783 \end{Bmatrix} \right) = \begin{Bmatrix} -0.1414 \\ -1.1049 \end{Bmatrix}$$

$$\left\{H^r_{Ci}\right\} = \frac{1}{20} \left(\begin{Bmatrix} -5.4373 \\ -63.9116 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -31.9558 \end{Bmatrix} \right) = \begin{Bmatrix} -0.2719 \\ -4.7934 \end{Bmatrix}$$

$$\left[h^r_{Ri}\right] = 0.0405 \left(\begin{bmatrix} -.0717 & .0662 \\ .1534 & .3032 \end{bmatrix} - \begin{bmatrix} .1517 & .1979 \\ -.0752 & -.1487 \end{bmatrix} \right)$$

$$[h^r_{Ri}] = \begin{bmatrix} -.0091 & -.0053 \\ .0093 & .0183 \end{bmatrix}$$

$$[h^r_{Li}] = 0.0405 \left(\begin{bmatrix} 0.0035 & 0.2148 \\ 0.1517 & 0.1979 \end{bmatrix} - \begin{bmatrix} -.0017 & -.1053 \\ -.0717 & 0.0662 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.0002 & 0.0130 \\ 0.0091 & 0.0053 \end{bmatrix}$$

$$[h^r_{Ci}] = \frac{1}{20} \left(\begin{bmatrix} 0.0082 & 0.4990 \\ 0.4750 & 0.9389 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.2375 & 0.4695 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0.0004 & 0.0250 \\ 0.0356 & 0.0704 \end{bmatrix}$$

End moments are computed from Eqs. (3.22) and (3.23).

$$\{M^e_{Ri}\} = \begin{Bmatrix} 0 \\ 104.04 \end{Bmatrix} + \begin{Bmatrix} 7.7783 \\ -20.6381 \end{Bmatrix} = \begin{Bmatrix} 7.7783 \\ 83.4019 \end{Bmatrix}$$

$$\{M^e_{Li}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -2.3410 \\ -19.4902 \end{Bmatrix} = \begin{Bmatrix} -2.3410 \\ -19.4902 \end{Bmatrix}$$

$$\{M^e_{Ci}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -5.4373 \\ -63.9116 \end{Bmatrix} = \begin{Bmatrix} -5.4373 \\ -63.9116 \end{Bmatrix}$$

$$[m^e_{Ri}] = \begin{bmatrix} .78 & -.78 \\ 0 & .78 \end{bmatrix} + \begin{bmatrix} -.0717 & .0662 \\ .1534 & .3032 \end{bmatrix} = \begin{bmatrix} .7083 & -.7138 \\ .1534 & 1.0832 \end{bmatrix}$$

$$\left[m^e_{Li}\right] = \begin{bmatrix} .78 & .0 \\ -.78 & .78 \end{bmatrix} + \begin{bmatrix} .0035 & .2148 \\ .1517 & .1979 \end{bmatrix} = \begin{bmatrix} .7835 & .2148 \\ -.6283 & .9779 \end{bmatrix}$$

$$\left[m^e_{Ci}\right] = \begin{bmatrix} -1.5 & .0 \\ 0. & -3.00 \end{bmatrix} + \begin{bmatrix} .0082 & .4990 \\ .4750 & .9389 \end{bmatrix} = \begin{bmatrix} -1.4918 & 0.4990 \\ 0.4750 & -2.0611 \end{bmatrix}$$

End thrusts are computed from Eqs. (3.24) and (3.25).

$$\left\{H^e_{Ri}\right\} = \begin{Bmatrix} 0 \\ 34.512 \end{Bmatrix} + \begin{Bmatrix} 1.1049 \\ -1.2463 \end{Bmatrix} = \begin{Bmatrix} 1.1049 \\ 33.2657 \end{Bmatrix}$$

$$\left\{H^e_{Li}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -0.1414 \\ -1.1049 \end{Bmatrix} = \begin{Bmatrix} -0.1414 \\ -1.1049 \end{Bmatrix}$$

$$\left\{H^e_{Ci}\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -0.2719 \\ -4.7934 \end{Bmatrix} = \begin{Bmatrix} -0.2719 \\ -4.7934 \end{Bmatrix}$$

$$\left[h^e_{Ri}\right] = \begin{bmatrix} .0735 & -.0735 \\ .0 & .0735 \end{bmatrix} + \begin{bmatrix} -.0091 & -.0053 \\ .0093 & .0183 \end{bmatrix} = \begin{bmatrix} .0644 & -.0788 \\ .0093 & .0918 \end{bmatrix}$$

$$\left[h^e_{Li}\right] = \begin{bmatrix} .0735 & .0 \\ -.0735 & .0735 \end{bmatrix} + \begin{bmatrix} 0.0002 & .0130 \\ 0.0091 & .0053 \end{bmatrix} = \begin{bmatrix} .0737 & .0130 \\ .0664 & .0788 \end{bmatrix}$$

$$\left[h^e_{Ci}\right] = \begin{bmatrix} 0.0750 & 0 \\ 0 & .3000 \end{bmatrix} + \begin{bmatrix} -.0004 & -.0250 \\ -.0356 & -.0704 \end{bmatrix} = \begin{bmatrix} .0746 & -.0250 \\ -.0356 & .2296 \end{bmatrix}$$

(5b) Unbalanced thrusts at joints are computed from Eqs. (3.26) and (3.27),

$$\left\{H_i^u\right\}_o = \begin{Bmatrix} 1.1049 \\ 33.2657 \end{Bmatrix} + \begin{Bmatrix} -0.1414 \\ -1.1049 \end{Bmatrix} + \begin{Bmatrix} 0.2719 \\ 4.7934 \end{Bmatrix} = \begin{Bmatrix} 1.2354 \\ 36.9542 \end{Bmatrix}$$

$$\left[h_i^u\right]_o = \begin{bmatrix} .0644 & -.0788 \\ .0093 & .0918 \end{bmatrix} + \begin{bmatrix} 0.0737 & .0130 \\ -.0644 & .0788 \end{bmatrix} + \begin{bmatrix} 0.0746 & -.0250 \\ -.0356 & 0.2296 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2127 & -.0908 \\ -.0908 & 0.4002 \end{bmatrix}$$

from which, the equilibrium equations at joints are written as

$$\begin{bmatrix} 0.2127 & -.0908 \\ -.0908 & 0.4002 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -1.2354 \\ -36.9542 \end{Bmatrix}$$

(5c) The solution of this matrix equation yields

$u_1 = -50.0702$ and $u_2 = -103.7021$

(6) Computation of the final moments and final thrusts

The final moments are obtained from Eqs. (3.29)

$$\begin{Bmatrix} M_{R1} \\ M_{R2} \end{Bmatrix} = \begin{Bmatrix} 7.7783 \\ 83.4019 \end{Bmatrix} + \begin{bmatrix} 0.7083 & -0.7138 \\ 0.1534 & 1.0832 \end{bmatrix} \begin{Bmatrix} -50.0702 \\ -103.7021 \end{Bmatrix} = \begin{Bmatrix} 46.3381 \\ -36.6078 \end{Bmatrix}$$

$$\begin{Bmatrix} M_{L1} \\ M_{L2} \end{Bmatrix} = \begin{Bmatrix} -2.3410 \\ -19.4902 \end{Bmatrix} + \begin{bmatrix} 0.7835 & 0.2148 \\ -0.6283 & 0.9779 \end{bmatrix} \begin{Bmatrix} -50.0702 \\ -103.7021 \end{Bmatrix} = \begin{Bmatrix} -63.8513 \\ -89.4348 \end{Bmatrix}$$

$$\begin{Bmatrix} M_{C1} \\ M_{C2} \end{Bmatrix} = \begin{Bmatrix} -5.4373 \\ -63.9116 \end{Bmatrix} + \begin{bmatrix} -1.4918 & 0.4990 \\ 0.4750 & -2.0611 \end{bmatrix} \begin{Bmatrix} -50.0702 \\ -103.7021 \end{Bmatrix} = \begin{Bmatrix} 17.5132 \\ 126.0426 \end{Bmatrix}$$

and the final thrusts are given by Eqs. (3.30).

$$\begin{Bmatrix} H_{R1} \\ H_{R2} \end{Bmatrix} = \begin{Bmatrix} 1.1049 \\ 33.2657 \end{Bmatrix} + \begin{bmatrix} 0.0644 & -.0788 \\ 0.0093 & 0.0918 \end{bmatrix} \begin{Bmatrix} -50.0702 \\ -103.7021 \end{Bmatrix} = \begin{Bmatrix} 6.0523 \\ 23.2831 \end{Bmatrix}$$

$$\begin{Bmatrix} H_{L1} \\ H_{L2} \end{Bmatrix} = \begin{Bmatrix} -0.1414 \\ -1.1049 \end{Bmatrix} + \begin{bmatrix} 0.0737 & 0.0130 \\ 0.0664 & 0.0788 \end{bmatrix} \begin{Bmatrix} -50.0702 \\ -103.7021 \end{Bmatrix} = \begin{Bmatrix} -5.1766 \\ -6.0523 \end{Bmatrix}$$

$$\begin{Bmatrix} H_{C1} \\ H_{C2} \end{Bmatrix} = \begin{Bmatrix} -0.2719 \\ -4.7934 \end{Bmatrix} + \begin{bmatrix} -.0746 & 0.0250 \\ 0.0356 & -.2296 \end{bmatrix} \begin{Bmatrix} -50.0702 \\ -103.7021 \end{Bmatrix} = \begin{Bmatrix} 0.8757 \\ 17.2308 \end{Bmatrix}$$

4.3.2 Generalized Infinite Matrix Series Method

The same continuous arch system which were analyzed by the restrained method in the preceding section will now be solved by using the generalized method.

Solution

(1) Various stiffness factors of segmental arches are taken from graphs 16, 18, 19 and 20.

$$k11_{Ri} = k11_{Li} = 9.95 \frac{EI_o}{Lr^2} = 9.95 \frac{100}{60*15*15} = 0.0735$$

$$k33_{Ri} = k33_{Li} = 12.9167$$

$$k13_{Ri} = k13_{Li} = 7.02 \frac{EI_o}{Lr} = 7.02 \frac{100}{60*15} = 0.7800$$

$$k23 = k23 = -3.95 \frac{EI_o}{L^2} = -3.95 \frac{100}{60*60} = -0.1097$$

For column 11′, the modified stiffness values are

$$k11_{C1} = \frac{3EI_c}{L_{C1}^3} = \frac{3*200}{20*20*20} = 0.0750$$

$$k33_{C1} = 30$$

$$k13_{C1} = \frac{-3EI_c}{L_{C1}^2} = -\frac{3*200}{20*20} = -1.5$$

and for column 22′,

$$k11_{C2} = \frac{12EI_c}{L_{C2}^3} = \frac{12*200}{20*20*20} = 0.3000$$

$$k33_{C2} = 40$$

$$k13_{C2} = \frac{-6EI_c}{L_{C2}^2} = -\frac{6*200}{20*20} = -3.0$$

(2) Distribution and transmission factors

Distribution factors for various members framing into joints 1 and 2 are calculated from Eqs. (2.33.d).

$$\sum k11_1 = .0735 + .0735 + .0750 = 0.2220$$

$$\sum k11_2 = .0735 + .0735 + .3000 = 0.4470$$

$$\sum k33_1 = 12.9167 + 12.9167 + 30. = 55.8333$$

$$\sum k33_2 = 12.9167 + 12.9167 + 40. = 65.8333$$

$$\sum k13_1 = 0.78 + 0.78 - 1.5 = 0.0600$$

$$\sum k13_2 = 0.78 + 0.78 - 3.0 = -1.4400$$

$$\sum k11_1 * \sum k33_1 - \sum k13_1 * \sum k31_1 = 0.222*55.8333 - 0.06*0.06 = 12.3914$$

$$\sum k11_2 * \sum k33_2 - \sum k13_2 * \sum k31_2 = 0.447*65.8333 - 1.44*1.44 = 27.3539$$

$d11_{R1} = d11_{L1}$	$= -(.0735*55.8333 - .78*.06)/12.3914$	$= -0.3274$
$d11_{C1}$	$= -(.075*55.8333 + 1.5*0.06)/12.3914$	$= -0.3453$
$d11_{R2} = d11_{L2}$	$= -(.0735*65.8333 + .78*1.44)/27.3539$	$= -0.2179$
$d11_{C2}$	$= -(0.3*65.8333 - 3.*1.44)/27.3539$	$= -0.5641$
$d13_{R1} = d13_{L1}$	$= -(.78*.222 - .0735*.06)/12.3914$	$= -0.0136$
$d13_{C1}$	$= -(1.5*.222 - .075*.06)/12.3914$	$= -0.0272$
$d13_{R2} = d13_{L2}$	$= -(.78*.447 + .0735*1.44)/27.3539$	$= -0.0166$
$d13_{C2}$	$= -(3.*.447 + .3*1.44)/27.3539$	$= -0.0332$
$d31_{R1} = d31_{L1}$	$= -(.78*55.8333 - 12.9167*.06)/12.3914$	$= -3.4526$
$d31_{C1}$	$= -(-1.5*55.8333 - 30.*.06)/12.3914$	$= 6.9052$
$d31_{R2} = d31_{L2}$	$= -(.78*65.8333 + 12.9167*1.44)/27.3539$	$= -2.5575$
$d31_{C2}$	$= -(-3.*65.8333 - 40.*1.44)/27.3539$	$= 5.1149$
$d33_{R1} = d33_{L1}$	$= -(12.9167*.222 - .78*.06)/12.3914$	$= -0.2276$
$d33_{C1}$	$= -(30.*.222 + 1.5*.06)/12.3914$	$= -0.5548$
$d33_{R2} = d33_{L2}$	$= -(12.9167*.447 + .78*1.44)/27.3539$	$= -0.2521$
$d33_{C2}$	$= -(30.*.447 + 3.*1.44)/27.3539$	$= -0.4957$

Transmission factors are given by Eqs. (2.35.c).

$t11_{R1}$	$= - d11_{R1}$	$= 0.3274$
$t13_{R1}$	$= - d13_{R1}$	$= 0.0136$
$t31_{R1}$	$= 3.4526 - (0.06*0.1097*60)/12.3914$	$= 3.4845$
$t33_{R1}$	$= 0.2276 - (0.222*.1097*60)/12.3914$	$= 0.1097$
$t11_{L2}$	$= - d11_{L2}$	$= 0.2179$
$t13_{L2}$	$= - d13_{L2}$	$= 0.0136$
$t31_{L2}$	$= 2.5575 - (1.44*0.1097*60)/27.3539$	$= 2.2109$
$t33_{L2}$	$= 0.2521 - (.447*0.1097*60)/27.3539$	$= 0.1446$

Distribution and transposed transmission matrices are compiled from the preceding calculated values,

$$[D_{Ri}] = \begin{bmatrix} -0.3274 & -0.0136 & 0 & 0 \\ -3.4526 & -0.2276 & 0 & 0 \\ 0 & 0 & -0.2179 & -0.0166 \\ 0 & 0 & -2.5575 & -0.2521 \end{bmatrix}$$

$$[D_{Li}] = \begin{bmatrix} -0.3274 & -0.0136 & 0 & 0 \\ -3.4526 & -0.2276 & 0 & 0 \\ 0 & 0 & -0.2179 & -0.0166 \\ 0 & 0 & -2.5575 & -0.2521 \end{bmatrix}$$

$$[D_{Ci}] = \begin{bmatrix} -0.3453 & 0.0272 & 0 & 0 \\ 6.9052 & -0.5447 & 0 & 0 \\ 0 & 0 & -0.5641 & 0.0332 \\ 0 & 0 & 5.1149 & -0.4957 \end{bmatrix}$$

$$[T'_{R1}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.3274 & 0.0136 & 0 & 0 \\ 3.4845 & 0.1097 & 0 & 0 \end{bmatrix}$$

$$[T'_{L2}] \quad \begin{bmatrix} 0 & 0 & 0.2197 & 0.0166 \\ 0 & 0 & 2.2109 & 0.1446 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[T_i'] = \begin{bmatrix} 0 & 0 & 0.2179 & 0.0166 \\ 0 & 0 & 2.2109 & 0.1446 \\ 0.3274 & 0.0136 & 0 & 0 \\ 3.4845 & 0.1097 & 0 & 0 \end{bmatrix}$$

(3) Fixed-end moments and fixed-end thrusts

These values are the same as used in the restrained method,

$$M^f_{R1} = M^f_{L1} = M^f_{L2} = M^f_{C1} = M^f_{C2} = 0$$

$$M^f_{R2} = 104.04 \text{ ft-kips}$$

$$H^f_{R1} = H^f_{L1} = H^f_{L2} = H^f_{C1} = H^f_{C2} = 0$$

$$H^f_{R2} = 34.512 \text{ kips}$$

and these fixed-end values can be written in the following generalized fixed-end force vector:

$$\{F^f_{Ri}\} = \begin{Bmatrix} 0 \\ 0 \\ 34.512 \\ 104.04 \end{Bmatrix}, \quad \{F^f_{Li}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \{F^f_{Ci}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

(4) Computation of unbalanced generalized forces

The initial unbalanced generalized force vector is

$$\{F^u_i\}_o = \begin{Bmatrix} 0 \\ 0 \\ 34.512 \\ 104.04 \end{Bmatrix}$$

The sum of all generalized forces relaxed at joints is given by the summation formula of the geometric matrix series in $[T']$, Eq. (3.40),

$$\left\{F_i^u\right\}_s =$$

$$\begin{bmatrix} 0 & 0 & 0.2179 & .0166 \\ 0 & 0 & 2.2109 & .1446 \\ 0.3274 & .0136 & 0 & 0 \\ 3.4845 & .1097 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -0.2179 & -.0166 \\ 0 & 1 & -2.2109 & -.1446 \\ -0.3274 & -.0136 & 1 & 0 \\ -3.4845 & -.1097 & 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ 34.512 \\ 104.04 \end{Bmatrix}$$

$$= \begin{bmatrix} 0.1566 & .0058 & .2649 & .0201 \\ 1.4881 & .0557 & 2.6582 & .1773 \\ 0.3989 & .0163 & .1229 & .0090 \\ 4.1934 & .1360 & 1.2146 & .0893 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 34.512 \\ 104.04 \end{Bmatrix} = \begin{matrix} 11.2289 \\ 110.1893 \\ 5.1766 \\ 51.2131 \end{matrix}$$

and the total unbalanced generalized forces are the sums of the generalized fixed-end forces and the corresponding unbalanced generalized forces,

$$\left\{F_i^u\right\}_t = \begin{Bmatrix} 0 \\ 0 \\ 34.512 \\ 104.04 \end{Bmatrix} + \begin{Bmatrix} 11.2289 \\ 110.1893 \\ 5.1766 \\ 51.2131 \end{Bmatrix} = \begin{Bmatrix} 11.2289 \\ 110.1893 \\ 39.6886 \\ 155.2531 \end{Bmatrix}$$

(6) Computation of the final moments and final thrusts

Final moments and final thrusts at joints are computed from Eqs. (3.42).

$$\{F_{Ri}\} = \begin{bmatrix} -0.3274 & -.0136 & 0.2179 & .0166 \\ -3.4526 & -.2276 & 2.2109 & .1446 \\ 0 & 0 & -.2179 & -.0166 \\ 0 & 0 & -2.5575 & -.2521 \end{bmatrix} \begin{Bmatrix} 11.2289 \\ 110.1893 \\ 39.6886 \\ 155.2531 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 34.512 \\ 104.04 \end{Bmatrix}$$

$$\begin{Bmatrix} H_{R1} \\ M_{R1} \\ H_{R2} \\ M_{R2} \end{Bmatrix} = \begin{Bmatrix} 6.0523 \\ 46.3381 \\ 23.2831 \\ -36.6078 \end{Bmatrix}$$

$$\{F_{Li}\} = \begin{bmatrix} -0.3274 & -.0136 & 0 & 0 \\ -3.4526 & -.2276 & 0 & 0 \\ 0.3274 & .0136 & -.2179 & -.0166 \\ 3.4845 & .1097 & -2.5575 & -.2521 \end{bmatrix} \begin{Bmatrix} 11.2289 \\ 110.1893 \\ 39.6886 \\ 155.2531 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} H_{L1} \\ M_{L1} \\ H_{L2} \\ M_{L2} \end{Bmatrix} = \begin{Bmatrix} -5.1766 \\ -63.8513 \\ -6.0523 \\ -89.4347 \end{Bmatrix}$$

$$\{F_{Ci}\} = \begin{bmatrix} -0.3453 & .0272 & 0 & 0 \\ 6.9052 & -.5447 & 0 & 0 \\ 0 & 0 & -0.5641 & .0332 \\ 0 & 0 & 5.1149 & -.4957 \end{bmatrix} \begin{Bmatrix} 11.2289 \\ 110.1893 \\ 39.6886 \\ 155.2531 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} H_{C1} \\ M_{C1} \\ H_{C2} \\ M_{C2} \end{Bmatrix} = \begin{Bmatrix} -0.8757 \\ 17.5123 \\ -17.2308 \\ 126.0424 \end{Bmatrix}$$

4.4 Approximations

The same structure shown in Fig. 9 which was analyzed by exact solutions will now be solved by approximate technique. Approximations can be made by taking partial sums of the infinite matrix series in $[T']$.

The successive powers of $[T']$ are computed as follows:

$$[T'] = \begin{bmatrix} 0 & 0 & 0.2179 & 0.0166 \\ 0 & 0 & 2.2109 & 0.1446 \\ 0.3274 & 0.0136 & 0 & 0 \\ 3.4845 & 0.1097 & 0 & 0 \end{bmatrix}$$

$$[T']^2 = \begin{bmatrix} 0.1292 & 0.0048 & 0 & 0 \\ 1.2275 & 0.0460 & 0 & 0 \\ 0 & 0 & 0.1015 & 0.0074 \\ 0 & 0 & 1.0019 & 0.0738 \end{bmatrix}$$

$$[T']^3 = \begin{bmatrix} 0 & 0 & 0.0388 & 0.0028 \\ 0 & 0 & 0.3691 & 0.0270 \\ 0.0590 & 0.0022 & 0 & 0 \\ 0.5850 & 0.0217 & 0 & 0 \end{bmatrix}$$

$$[T']^4 = \begin{bmatrix} 0.0226 & 0.0008 & 0 & 0 \\ 0.2151 & 0.0080 & 0 & 0 \\ 0 & 0 & 0.0177 & 0.0013 \\ 0 & 0 & 0.1755 & 0.0129 \end{bmatrix}$$

$$[T']^5 = \begin{bmatrix} 0 & 0 & 0.0068 & 0.0005 \\ 0 & 0 & 0.0645 & 0.0047 \\ 0.0103 & 0.0004 & 0 & 0 \\ 0.1023 & 0.0038 & 0 & 0 \end{bmatrix}$$

$$[T']^6 = \begin{bmatrix} 0.0040 & 0.0002 & 0 & 0 \\ 0.0376 & 0.0014 & 0 & 0 \\ 0 & 0 & 0.0031 & 0.0002 \\ 0 & 0 & 0.0307 & 0.0023 \end{bmatrix}$$

The partial sum $[S_4]$ of terms up to and including the fourth power of $[T']$ is

$$[S_4] = \begin{bmatrix} 1.1518 & 0.0056 & 0.2570 & 0.0195 \\ 1.4426 & 1.0540 & 2.5800 & 0.1716 \\ 0.3864 & 0.0158 & 1.1192 & 0.0087 \\ 4.0694 & 0.1314 & 1.1774 & 0.0866 \end{bmatrix}$$

From this sum, the final generalized forces may be obtained by the rule of distribution and carry-over, using Eqs. (3.42). These forces are

$$\{F_{Ri}\} = \begin{Bmatrix} 6.0194 \\ 46.1019 \\ 23.3436 \\ -35.8081 \end{Bmatrix}, \quad \{F_{Li}\} = \begin{Bmatrix} -5.1489 \\ -63.5112 \\ -6.0194 \\ -88.9091 \end{Bmatrix}, \quad \{F_{Ci}\} = \begin{Bmatrix} -0.8705 \\ 17.4094 \\ -17.1937 \\ 126.0104 \end{Bmatrix}$$

If the partial sum were taken up to the sixth power of $[T']$, the final forces would be

$$\{F_{Ri}\} = \begin{Bmatrix} 6.0465 \\ 46.2969 \\ 23.2937 \\ -36.4680 \end{Bmatrix}, \quad \{F_{Li}\} = \begin{Bmatrix} -5.1718 \\ -63.7918 \\ -6.0465 \\ -89.3428 \end{Bmatrix}, \quad \{F_{Ci}\} = \begin{Bmatrix} -0.8748 \\ 17.4951 \\ -17.2243 \\ 126.0369 \end{Bmatrix}$$

It is seen that these values are close to those of exact solutions.

4.5 Errors Committed in Stopping at Any Stage

From Eqs. (3.45) and (3.46), successive error matrices can be computed as follows:

$$[E_4] = \begin{bmatrix} 0 & 0 & 0.0068 & 0.0005 \\ 0 & 0 & 0.0645 & 0.0047 \\ 0.0103 & 0.0004 & 0 & 0 \\ 0.1023 & 0.0038 & 0 & 0 \end{bmatrix}$$

$$[E_5] = \begin{bmatrix} 0.0040 & 0.0002 & 0 & 0 \\ 0.0376 & 0.0014 & 0 & 0 \\ 0 & 0 & 0.0031 & 0.0002 \\ 0 & 0 & 0.0307 & 0.0032 \end{bmatrix}$$

$$[E_6] = \begin{bmatrix} 0 & 0 & 0.0012 & 0.0001 \\ 0 & 0 & 0.0113 & 0.0008 \\ 0.0018 & 0.0001 & 0 & 0 \\ 0.0179 & 0.0007 & 0 & 0 \end{bmatrix}$$

$$[E_7] = \begin{bmatrix} 0.0007 & 0.0000 & 0 & 0 \\ 0.0066 & 0.0003 & 0 & 0 \\ 0 & 0 & 0.0005 & 0.0000 \\ 0 & 0 & 0.0054 & 0.0004 \end{bmatrix}$$

$$[E_8] = \begin{bmatrix} 0 & 0 & 0.0002 & 0.0000 \\ 0 & 0 & 0.0020 & 0.0001 \\ 0.0003 & 0.0000 & 0 & 0 \\ 0.0031 & 0.0001 & 0 & 0 \end{bmatrix}$$

It is seen that at the end of the eighth cycle of release, elements of the error matrix are small enough to be neglected; the total error matrix is

$$[E_t] = \begin{bmatrix} 0.0047 & 0.0002 & 0.0082 & 0.0006 \\ 0.0442 & 0.0017 & 0.0778 & 0.0057 \\ 0.0124 & 0.0005 & 0.0037 & 0.0003 \\ 0.1233 & 0.0046 & 0.0361 & 0.0027 \end{bmatrix}$$

Error vectors of moment and thrust committed in stopping at the fourth release are computed from Eqs. (3.49)

$$\{e_{Ri}\} = \begin{Bmatrix} 0.0333 \\ 0.2429 \\ -0.0602 \\ -0.7952 \end{Bmatrix}, \quad \{e_{Li}\} = \begin{Bmatrix} -0.0269 \\ -0.3302 \\ -0.0333 \\ -0.5292 \end{Bmatrix}, \quad \{e_{Ci}\} = \begin{Bmatrix} 0.0050 \\ -0.1009 \\ 0.0369 \\ -0.0319 \end{Bmatrix}$$

4.6 Estimate of Rates of Convergence

In the restrained infinite matrix series method, it is evident that the norm of the transmission matrix is less than unity and the series in $[T']$ is convergent. The rate of convergence can be estimated by using inequality (B.12) in Appendix B.

For the given example problem,

$$\frac{\|T'\|^{k+1}}{1 - \|T'\|} = \frac{(0.1487)^{k+1}}{0.8513} = (1.1747)(0.1487)^{k+1}$$

It is seen that the balancing process converges quickly.

In the generalized infinite matrix series, the third norm of the transmission matrix is

$$\|T'\|_3 = 5.7456,$$

which is greater than unity, and it is not clear wether the series will converge or not. However, from Eq. (3.41) which is based on a Sylvester theorem, the largest eigevalue is approximated as

$$\lambda = \left(\frac{0.0226}{0.1292}\right)^{0.5} = 0.4182$$

This value is less than unity, then $[T']^k \rightarrow 0$, and the process is convergent.

Third norms of error matrices are given by Eq. (B.8).

$$\|E_4\|_3 = 0.1217$$

$$\|E_5\|_3 = 0.0489$$

$$\|E_6\|_3 = 0.0213$$

$$\|E_7\|_3 = 0.0085$$

$$\|E_8\|_3 = 0.0037$$

This promises merely that the balancing process is gradually convergent.

From the above determination of rates of convergence, one can estimate how many steps must be taken to obtain a given accuracy.

4.7 Computer Solution

4.7.1 Computer Programs

Computer programs for the analysis of continuous curvilinear structures by infinite matrix series methods are written to perform the matrix operations which consist of several matrix additions, matrix multiplications, matrix inversion and solution of systems of equations. Symbols used in these computer programs are similar to those in the list of symbols. The listing of these programs is included in Appendix D.

Only input and output data are described in the following numerical examples.

The input data include stiffness factors and geometry of segmental arches and columns which define the configuration of the continuous system and external forces which describe the applied forces at joints. The programs can handle various loading conditions such as concentrated loads, uniform loads, or support settlements and their combinations.

The computer output gives directly the final moments and thrusts.

It is important that the final output data be examined for its correctness before it is accepted for use. This may be done by checking to see whether the two resolution equations of equilibrium are satisfied at each interior joint. At the supports, the resolution equations are used to determine the reactions. Finally, the three equilibrium conditions for the entire continuous system as a free body are verified.

Three numerical examples are chosen to show the validity of the generalized infinite matrix series method.

4.7.2 Numerical Example 1 – Three Bay Portal Frame

Many examples are available in previous publications, and for comparison of methods it is desirable that one of these should be used here.

The continuous arch frame shown in Fig. 10 has been used elsewhere[5,13,14,15,17]. The frame has constant flexural rigidity EI in all members. Two loading conditions are considered, the uniform load first on span 12 and then on span 23.

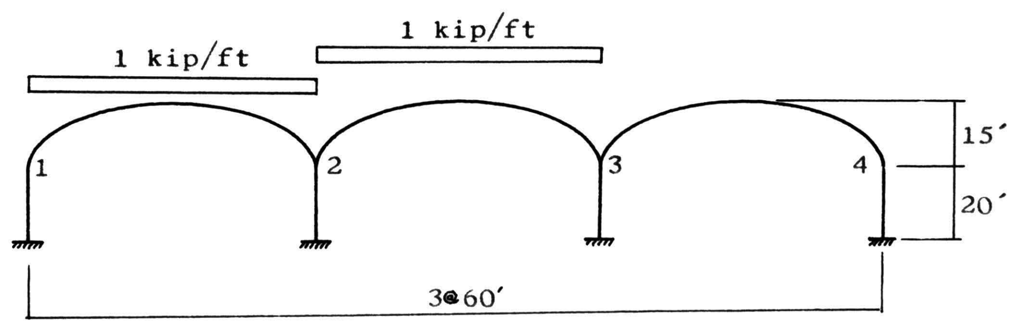


Fig. 10. Three-Bay Portal Frame

For reason of comparison, various stiffness values and fixed-end moments and fixed-end thrusts are taken identically with those used by Maugh[5]. It should be mentioned, however, that these values appear to be slightly inexact.

Constants for segmental arches and columns are

$$k11_{Ri} = k11_{Li} = 0.77$$

$$k11_{Ci} = 1.50$$

$$k33_{Ri} = k33_{Li} = 132.40$$

$$k33_{Ci} = 200.$$

$$k13_{Ri} = k13_{Li} = 8.20$$

$$k13_{Ci} = -15.00$$

$$k23_{Ri} = k23_{Li} = -1.063$$

$$k23_{Ci} = 0$$

$$L = 60. \quad \text{and} \quad L_{Ci} = 20.$$

Fixed-end moments and thrusts are expressed in following matrices:

$$[F^f_{Ri}] = \begin{bmatrix} 36 & 0 \\ 126 & 0 \\ 0 & 36 \\ 0 & 126 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad [F^f_{Li}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -36 & 0 \\ -126 & 0 \\ 0 & -36 \\ 0 & -126 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad [F^f_{Ci}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following computer output solution agrees with the results published by the previously mentioned investigators.

$$\begin{bmatrix} H_{R1} \\ M_{R1} \\ H_{R2} \\ M_{R2} \\ H_{R3} \\ M_{R3} \\ H_{R4} \\ M_{R4} \end{bmatrix} = \begin{bmatrix} 14.6397 & 2.1236 \\ -122.7449 & 0.0629 \\ 2.4059 & 16.0274 \\ 29.4530 & -91.4209 \\ 0.9136 & 2.1236 \\ 21.6468 & 38.6587 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} H_{L1} \\ M_{L1} \\ H_{L2} \\ M_{L2} \\ H_{L3} \\ M_{L3} \\ H_{L4} \\ M_{L4} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -14.6397 & -2.1236 \\ 92.7486 & -38.6586 \\ 2.4059 & -16.0274 \\ -16.6356 & 91.4209 \\ -0.9136 & -2.1236 \\ 1.1646 & -0.0629 \end{bmatrix}$$

$$\begin{bmatrix} H_{C1} \\ M_{C1} \\ H_{C2} \\ M_{C2} \\ H_{C3} \\ M_{C3} \\ H_{C4} \\ M_{C4} \end{bmatrix} = \begin{bmatrix} 14.6397 & -2.1236 \\ 122.7445 & -0.0626 \\ 12.2338 & -13.9038 \\ -122.2010 & 130.0792 \\ 1.4923 & 13.9038 \\ -5.0116 & 130.0791 \\ 0.9137 & 2.1235 \\ -1.1646 & 0.0626 \end{bmatrix}$$

4.7.3 Numerical Example 2 — Support Settlements

Consider a continuous arch frame of constant cross-section, shown in Fig. 11, as an example to illustrate the application of infinite matrix series methods for effects of support settlements. Each curved member is a semi-elliptical arch of 15 ft span and 5 ft rise. Ends of arch system are fixed. Interior columns at 1 and 2 have 12 ft and 6 ft height respectively and are fixed at base. The structure is analyzed for horizontal, vertical and rotational displacements at support 1'.

Given: $E = 30000$ ksi, $I = 1000$ in.^{4}

$u_{1'} = 1$ in., $v_{1'} = 1$ in., $\theta_{1'} = 0.01$ rad.

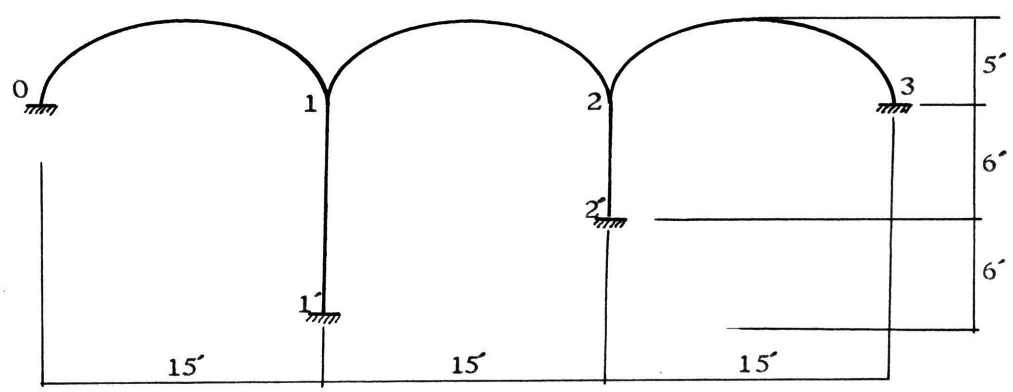


Fig. 11. Support Settlements of Continuous Arch Frame

Input data

Various stiffness factors for segmental arches and columns are

$$k11_{Ri} = k11_{Li} = \frac{9.108\ EI}{15*5*5} = 0.02428$$

$$k11_{C1} = \frac{12EI}{12*12*12} = 0.00694$$

$$k11_{C2} = \frac{12\ EI}{6*6*6} = 0.05556$$

$$k33_{Ri} = k33_{Li} = \frac{6.895\ EI}{15} = 0.33333$$

$$k33_{C1} = \frac{4\ EI}{12} = 0.33333$$

$$k33_{C2} = \frac{4EI}{6} = 0.66667$$

$$k13_{Ri} = k13_{Li} = \frac{6.278\ EI}{15*5} = 0.08370$$

$$k13_{C1} = \frac{-6EI}{12*12} = -0.04167$$

$$k13_{C2} = \frac{-6EI}{6*6} = -0.16667$$

$$k23_{Ri} = k23_{Li} = \frac{-3.570\ EI}{15*15} = -0.01587$$

$$k23_{C1} = k23_{C2} = 0$$

$$L = 15, \quad L_{C1} = L_{C2} = 6$$

The fixed-end moments and thrusts are computed as follows:

For the horizontal support settlement $u_{1'} = 1$ in.

$$H^f_{R1} = H^f_{L1} = H^f_{R2} = H^f_{L2} = H^f_{C2} = 0$$

$$H^f_{C1} = \frac{-12EI}{L^3_{C1}} u_{1'} = -\frac{12*30000*1000*1}{12*12*12*1728} = -120.5633 \text{ kips}$$

$$M^f_{R1} = M^f_{L1} = M^f_{R2} = M^f_{L2} = M^f_{C2} = 0$$

$$M^f_{C1} = \frac{6EI}{L^2_{C1}} u_{1'} = \frac{6*30000*1000*1}{12*12*1728} = 723.3796 \text{ ft-kips}$$

For the vertical support settlement $v_{1'} = 1$ in.

$$H^f_{R1} = H^f_{L1} = H^f_{C1} = H^f_{R2} = H^f_{L2} = H^f_{C2} = 0$$

$$M^f_{R1} = -M^f_{L1} = M^f_{L2} = k23\ v_{1'} = -3.5705\ \frac{EI}{L^2}\ v_{1'}$$

$$= \frac{-3.5705*30000*1000*1}{15*15*1728} = -275.5035$$

$$M^f_{R2} = M^f_{C1} = M^f_{C2} = 0$$

For rotational support settlement $\theta_{1'} = 0.01$ rad.

$$H^f_{R1} = H^f_{L1} = H^f_{R2} = H^f_{L2} = H^f_{C2} = 0$$

$$H^f_{C1} = \frac{-6EI}{L^2_{C1}}\ \theta_{1'} = \frac{-6*30000*1000*0.01}{12*12*144} = -86.8000 \text{ kips}$$

$$M^f_{R1} = M^f_{L1} = M^f_{R2} = M^f_{L2} = M^f_{C2} = 0$$

$$M^f_{C1} = \frac{2EI}{L_{C1}}\ \theta_{1'} = \frac{2*30000*1000*0.01}{12*144} = 347.2222 \text{ ft-kips}$$

These values can be expressed in the following loading matrices:

$$[F^f_{Ri}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -275.5035 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[F^f_{Li}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 275.5035 & 0 \\ 0 & 0 & 0 \\ 0 & -275.5035 & 0 \end{bmatrix}$$

$$[F^f_{Ci}] = \begin{bmatrix} -120.5633 & 0 & -86.8000 \\ 723.3796 & 0 & 347.2222 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The output matrices for the final thrusts and final moments shown below are obtained from the computer program for the analysis of continuous curvilinear structure by generalized infinite matrix series method, listed in Appendix D.

$$\begin{bmatrix} H_{R1} \\ M_{R1} \\ H_{R2} \\ M_{R2} \end{bmatrix} = \begin{bmatrix} 28.0510 & 6.3553 & 22.0183 \\ -186.0095 & -390.9687 & -82.4061 \\ 76.3773 & -14.4112 & 49.8606 \\ -409.7404 & 236.1770 & -216.9396 \end{bmatrix}$$

$$\begin{bmatrix} H_{L1} \\ M_{L1} \\ H_{L2} \\ M_{L2} \end{bmatrix} = \begin{bmatrix} 104.4282 & -8.0559 & 71.8786 \\ -621.2558 & 402.3596 & -315.5725 \\ -28.0510 & -6.3553 & -22.0183 \\ 246.9001 & -169.5741 & 122.0491 \end{bmatrix}$$

$$\begin{bmatrix} H_{C1} \\ M_{C1} \\ H_{C2} \\ M_{C2} \end{bmatrix} = \begin{bmatrix} -132.4791 & 1.7005 & -93.8983 \\ 807.2661 & -11.3911 & 397.9782 \\ -48.3264 & 20.7665 & -27.8423 \\ 162.8409 & -66.6031 & 94.8905 \end{bmatrix}$$

The problem of support settlements can be extended as a tool to solve the superstructure on a main simple arch as shown in Fig. 12.

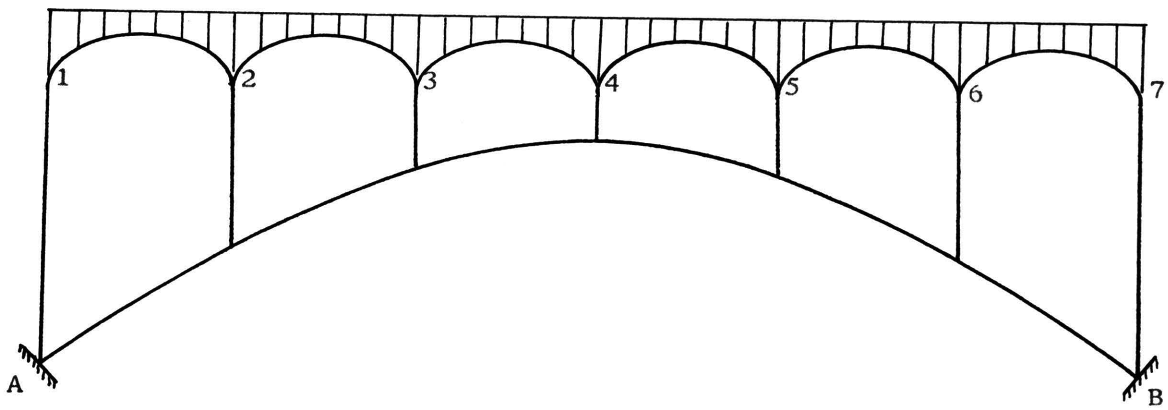


Fig. 12. Superstructure

Once the reactions at the column bases of the superstructure are determined for a given loading condition, the deflected curve of the main arch can be found and various support settlements for the superstructure are then known. These quantities may be considered as new loading conditions and the superstructure is analyzed by the infinite matrix series again. Then, it is possible to analyze the effect of the superstructure on the main arch by a trial-and-error technique.

4.7.4 Numerical Example 3 – Influence Lines of a Continuous Arch Frame

The computer solution may also be used to calculate influence lines for thrust and moment at the column base 2' of the model continuous arch frame as shown in Fig. 13. The frame is a three-bay portal model of constant cross-section (0.125 in. by 0.5 in.). Each bay is a uniform circular arch of 10 in. span and of 2.5 in. rise, resting on a pier of 3 in. height which is fixed at base.

Influence lines for thrust and moment may be drawn when a unit load is allowed to move along the horizontal span. Thus if incremental distances of one inch are used, the frame is considered to be subjected to 27 loading conditions. The loadings at the top of the columns have been omitted. Values of the fixed-end thrust and fixed-end moment due to unit load are taken from Table VII and VIII in Appendix A.

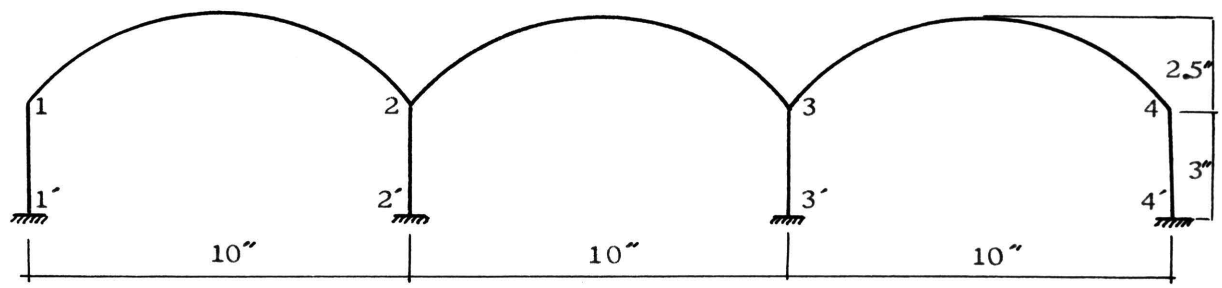


Fig. 13. Three-Bay Portal Frame Model

Thrusts at the base 2' are given by the computer output with a change of sign. These can be summarized in Table II.

Table II. Thrust Coefficients
for Influence Line $H_{2'}$ for the Three-Bay Portal Frame

X	1st span	2nd span	3rd span
0.1	-.0746	.0833	.0025
0.2	-.2106	.2251	.0125
0.3	-.3409	.3581	.0236
0.4	-.4315	.4482	.0324
0.5	-.4648	.4784	.0369
0.6	-.4355	.4440	.0364
0.7	-.3479	.3506	.0308
0.8	-.2189	.2162	.0209
0.9	-.0811	.0763	.0091

Moments at the base 2′ can be computed by using static equilibrium conditions, when moments and thrusts at the top of the column 22′ are known from the computer output. These values are obtained by simple calculations and arranged in Table III.

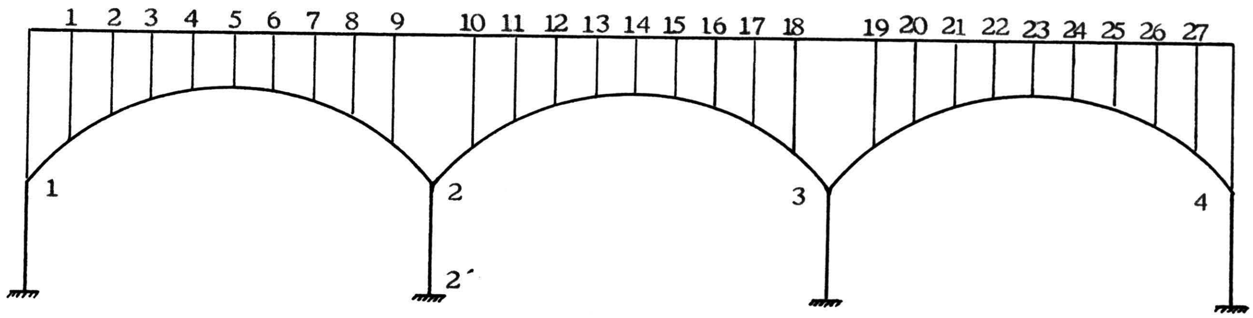
Table III. Moment Coefficients for Influence Line $M_{2'}$ for the Three-Bay Portal Frame

X	1^{st} span	2^{nd} span	3^{rd} span
0.1	-.1264	.1075	.0030
0.2	-.3305	.3232	.0226
0.3	-.5180	.5348	.0559
0.4	-.6417	.6852	.0852
0.5	-.6787	.7440	.1040
0.6	-.6236	.7016	.1080
0.7	-.4850	.5641	.0962
0.8	-.2929	.3573	.0698
0.9	-.0970	.1343	.0339

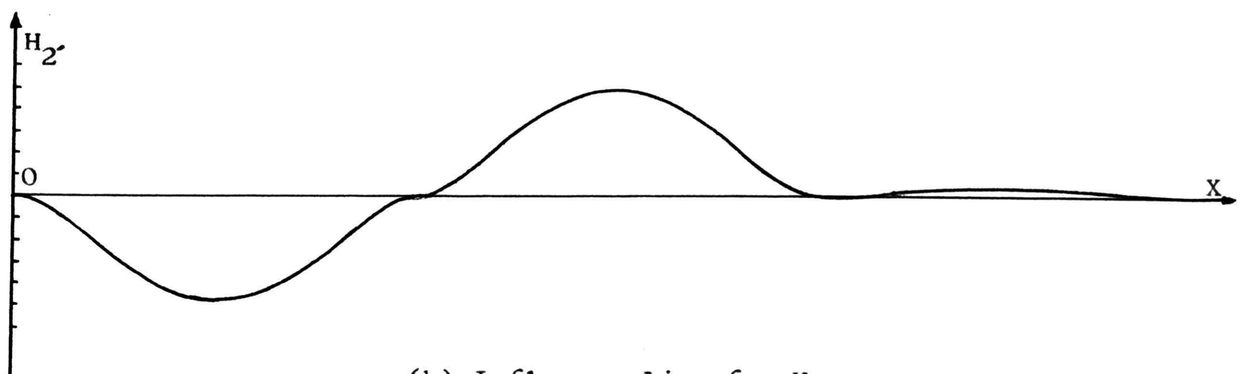
From values in Tables II and III, influence lines for thrust $H_{2'}$ and for moment $M_{2'}$ of the base 2′ can be drawn as shown in Figs. 14.b and 14.c.

4.8 Experimental Model Analysis

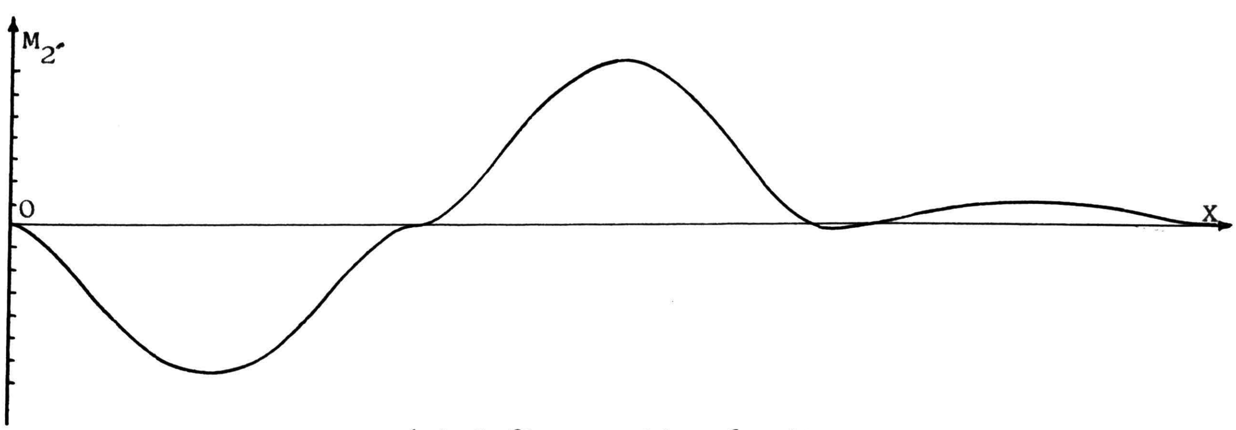
A model study based on the Muller-Breslau principle using the Beggs deformeter[31,32,33,34] was used to check the analytical results. The model was cut from an acrylic plastic sheet of 1/8 inch thickness. The frame has constant cross-section of 0.125 x 0.5 $in.^2$ in all members



(a) Continuous Arch Frame



(b) Influence Line for $H_{2'}$



(c) Influence Line for $M_{2'}$

Fig. 14. Influence Lines $H_{2'}$ and $M_{2'}$ for Three-Bay Portal Frame

and the geometry of the model is shown in Fig. 13.

Influence lines for thrust and moment at the base 2′ of the model were determined by taking displacement readings along the model for known amounts of deformation induced at 2′. Fig. 15 shows the set-up of the deformeter over the model of continuous arch frame.

The principle of the mechanical analysis of structural models and a description of the Beggs deformeter are given in Appendix C.

Data obtained from the test is shown in Tables IV and V together with the related calibration constants. These deflection readings divided by the corresponding calibration constant of the microscope, influence line coefficients for thrust and moment are recorded in columns 4, 7 and 10 in Tables IV and V.

A comparison of these experimental values with those obtained by analytical methods shown in Tables II and III indicates that the agreement is reasonably close.

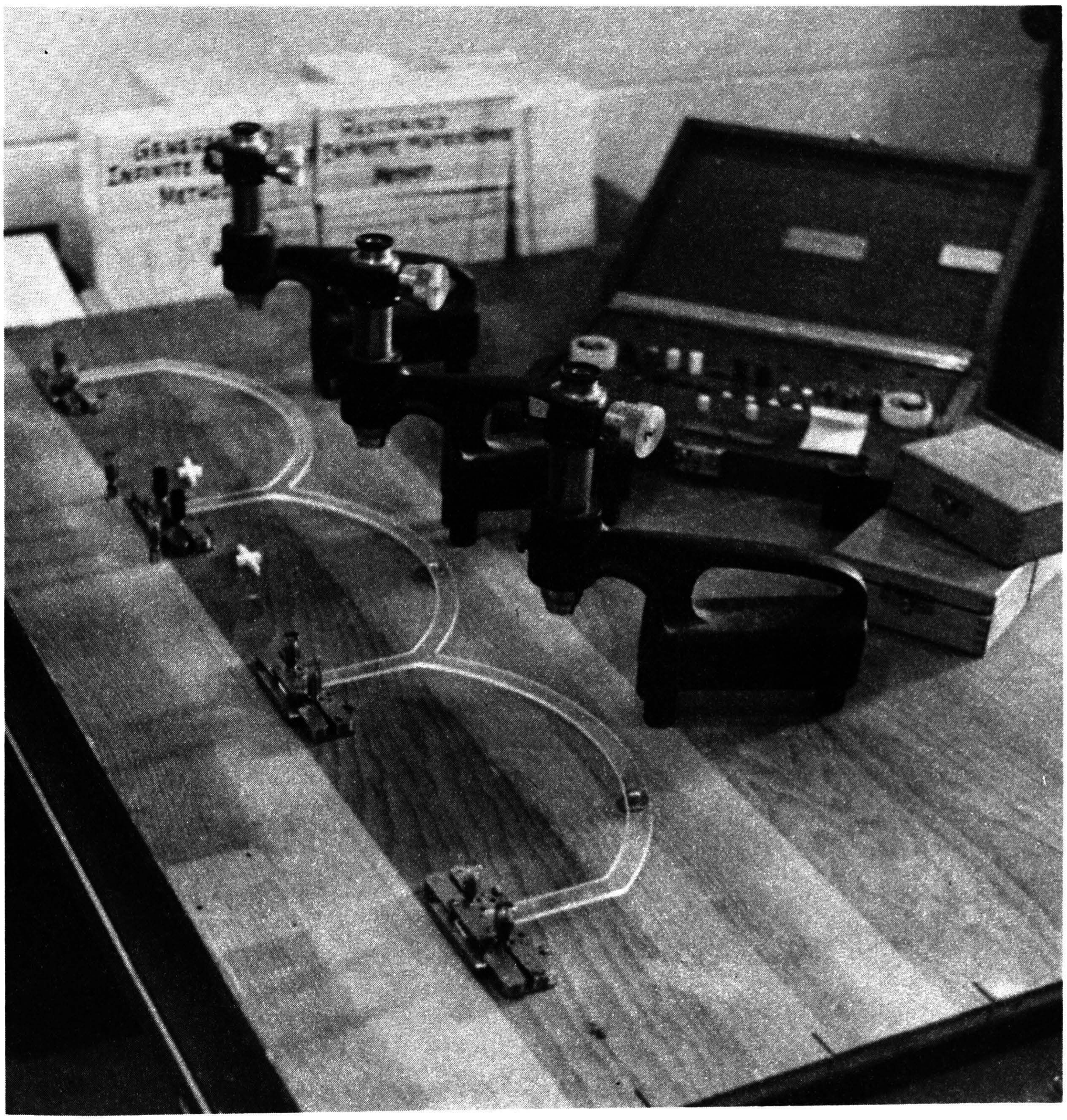


Fig. 15. Experimental Model Set-Up

Table IV. Experimental Data
for Influence Line H_2, for the Three-Bay Portal Model
(Beggs Deformeter Analysis)

calibration constant = 262.4

X	1st span			2nd span			3rd span		
	left	right	coeff.	left	right	coeff.	left	right	coeff.
0.1	402.0	420.0	-0.069	407.0	386.0	0.080	405.0	406.0	-0.004
0.2	405.0	455.5	-0.192	411.0	345.5	0.231	407.0	404.0	0.013
0.3	409.0	498.0	-0.338	409.5	315.0	0.360	410.0	404.0	0.026
0.4	402.5	514.0	-0.426	411.0	293.0	0.450	407.5	399.0	0.032
0.5	404.0	524.0	-0.458	409.0	283.0	0.481	408.0	398.0	0.038
0.6	399.0	510.5	-0.426	410.0	298.0	0.427	406.5	397.0	0.038
0.7	398.0	490.0	-0.351	408.0	316.0	0.351	407.0	399.0	0.030
0.8	409.0	467.0	-0.221	409.5	351.5	0.221	406.0	400.5	0.021
0.9	404.0	425.0	-0.080	407.5	386.5	0.080	405.0	403.0	0.008

Table V. Experimental Data
for Influence Line M_2, for the Three-Bay Portal Frame
(Beggs Deformeter Analysis)

calibration constant = 52.3

X	1st span			2nd span			3rd span		
	cw	c.cw	coeff.	cw	c.cw	coeff.	cw	c.cw	coeff.
0.1	401.0	408.0	-0.134	399.0	393.0	0.115	405.5	406.0	-0.001
0.2	407.0	424.0	-0.325	402.0	385.0	0.325	401.5	400.5	0.019
0.3	398.0	425.0	-0.517	404.0	376.0	0.535	402.5	399.5	0.058
0.4	402.0	435.5	-0.641	398.5	362.5	0.690	402.0	397.5	0.085
0.5	399.0	434.5	-0.680	405.0	366.0	0.745	399.0	393.5	0.105
0.6	402.0	435.5	-0.641	403.5	367.0	0.698	405.0	399.0	0.115
0.7	398.5	424.0	-0.488	400.5	370.5	0.575	403.5	398.5	0.095
0.8	405.5	421.5	-0.306	403.0	384.0	0.364	397.0	393.5	0.067
0.9	403.0	408.0	-0.095	405.5	398.5	0.134	398.0	399.5	0.029

cw = clockwise, c.cw = counter-clockwise

Chapter 5

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

5.1 Summary and Conclusions

An extension of the moment-distribution method has been presented for the analysis of continuous curvilinear structures by infinite matrix series methods which are relatively convenient for those types of frames incorporating curved members. Some of the features of these approaches may be summarized as follows:

(1) To analyze continuous arch frames, it is first necessary to know the flexural properties of each curved member. The general expressions for various stiffness factors as well as coefficients for fixed-end moment and fixed-end thrust with different loading conditions for three common types of symmetrical arch of circular, parabolic and semi-elliptical shapes with a wide range of rise to span ratios have been derived, graphed and tabulated. These stiffness and fixed-end reaction values are readily applied to the analysis of continuous arch systems and quickly solved by either of the two methods of infinite matrix series.

(2) The infinite matrix series approach is a superior method compared with the well-known moment-distribution method. It follows that the calculations involving successive approximations in the moment distribution method become unnecessary. In other words, they are balanced at one time only and the final forces may be calculated in a single stage of distribution and carry-over.

(3) Solutions obtained from these methods are direct and exact.

(4) Since these methods are based on matrix theory, the computation procedure for determining the final moments and thrusts is systematic, easy to understand, and suitable for digital electronic computer, particularly when the inversion of large matrices is required.

(5) Whenever electronic computation facilities are not available, useful approximate solutions may soon be obtained. Since the approximations totally eliminate the inverting of matrices, problems can be solved with a desk calculator.

(6) As a result, computational errors are easily detected and corrected. The support moments and thrusts obtained by distribution of moments and thrusts should be in exact equilibrium at each joint.

(7) In the restrained infinite matrix series method, the dimension of the transmission matrix is equal to the number of interior joints, which for most cases is relatively small. However, this method requires the effects of restraining forces and moments to be separately assessed, and this often leads to a tedious solution.

(8) In the generalized infinite matrix series method, translation as well as rotation of the joints is considered, so that the effect of sidesway is automatically taken into account. The dimension of the transmission matrix is twice as large as for the case of the restrained method. However, it is more elegant, more compact and essentially more straightforward than the restrained method. The generalized method is particularly useful where facilities exist for the inversion of a matrix.

(9) A major portion of hand computations involved is the evaluation of the distribution and transmission factors which, once found, do not change for different loading conditions since the fixed-end reactions are treated as a separate step. Thus, the methods are quite convenient for problems involving moving loads.

(10) The effects of support settlements on the continuous curvilinear structures have been also considered. It is possible to analyze a continuous superstructure on a main arch. Once the moments and thrusts at the bottom of the superstructure are known, the deflected curve of the main arch can be obtained. This particular problem then can be solved by trial-and-error technique.

5.2 Recommendations for Future Investigations

The infinite matrix series methods presented in this dissertation are restricted in application here to those plane structures in which the connecting joints are only able to rotate and displace horizontally. Further investigations, based on infinite matrix methods, can be extended to solve the following topics:

(1) The generalized infinite matrix series method can be extended quite easily to a general case where vertical displacements may also occur; in this case the further derivation of stiffness and transmission matrices of a segmental arch is needed.

(2) These methods of analysis may also be readily applied to other types of rigid jointed structures such as rectangular frames, gabled frames, continuous Vierendeel girders, grid-frames, unsymmetrical continuous arches, and other irregular types.

(3) The generalized infinite matrix series method could also be extended to cover the case of rigid jointed space curvilinear structures. In this case the procedure may be performed with simple extensions to allow for the torsional resistance. Stiffnesses of a segmental arch produced by unit linear and angular displacements can be expressed in a 6 by 6 matrix.

(4) The infinite matrix series methods can be used to analyze the effect of temperature on the continuous arch frames. The procedure consists of determining fixed-end moments and thrusts which are caused by temperature induced displacements, of distributing unbalanced moments and thrusts and then joint restraints can be found. It is desired to study the effects of temperature independent of any externally applied loads. The results can then be superimposed.

(5) The infinite matrix series methods are perhaps useful for the analysis of stability of arches.

BIBLIOGRAPHY

1. H. Cross, "Analysis of Continuous Frames by Distributing Fixed-End-Moments," Trans. ASCE, Vol. 96, 1932, Paper No. 1793.

2. D.E. Larson, Discussion (pp. 127-133) to H. Cross, "Analysis of Continuous Frames by Distributing Fixed-End Moments," Trans. ASCE, Vol. 96, 1932.

3. H. Cross and N.D. Morgan, Continuous Frames of Reinforced Concrete, John Wiley and Sons, Inc., New York, 1932, pp. 316-326.

4. A. Hrennikoff, "Analysis of Multiple Arches," Trans. ASCE, Vol. 101, 1936, Paper No. 1930.

5. L.C. Maugh, Statically Indeterminate Structures, John Wiley and Sons, Inc., New York, Second Edition, 1964, pp. 316-326.

6. V.A. Morgan, "An Exact Method of Analysis of Continuous Parabolic Arches," Concrete and Construction Engineering, Vol. 47, No. 11, Nov. 1952, pp. 343-352.

7. J. Michalos and D.D. Girton, "Continuous Arches on Elastic Piers," Proc. ASCE, Journal of the Structural Division, Vol. 81, Nov. 1955, Paper No. 827.

8. W.E. Riley, "Analysis of Continuous Arches on Flexible Piers," Journal of the American Concrete Institute, Vol. 28, No. 10, April 1957, pp. 999-1012.

9. J.J. Tuma, K.S. Havner and F. Hedges, "Analysis of Frames with Curved and Bent Members," Proc. ASCE, Journal of the Structural Division, Vol. 84, Sept. 1958, Paper No. 1764.

10. A.P. Hjort, "Design of Continuous Arches on Elastic Piers," Journal of the American Concrete Institute, Nov. 1932, pp. 143-147.

11. K.T. Fowler, "Slope Deflection Equations for Curved Members," Trans. ASCE, Vol. 116, 1951, Paper No. 2428.

12. J.I. Parcel and R.B.B. Moorman, Analysis of Statically Indeterminate Structures, John Wiley and Sons, Inc., Sixth Printing, 1967, pp. 408-430.

13. E. Lightfoot, "Generalized Slope-Deflection and Moment-Distribution Methods for the Analysis of Viaducts and Multi-Bay, Ridged Portal Frames," The Structural Engineer, Vol. 36, Jan. 1958, pp. 21-33.

14. T.M. Wang and D.C. Church, "Analysis of Continuous Arch Frames," Proc. ASCE, Journal of the Structural Division, Vol. 93, Oct. 1967, Paper No. 5526.

15. L.A. Beaufoy and A.F.S. Diwan, "Analysis of Continuous Structures by the Stiffness Factors Method," Quarterly Journal of Mechanics and Applied Mathematics, Vol. II, Sept. 1949, pp. 263-282.

16. E. Lightfoot, Moment Distribution - A Rapid Method of Analysis for Rigid-Jointed Structures, John Wiley and Sons Inc., 1961, pp. 252-288.

17. F.P. Wiesinger, S.L. Lee and D.L. Guell, "Analysis of Arch Frames," Proc. ASCE, Journal of the Structural Division, Vol. 89, Oct. 1963, Paper No. 3660.

18. H. Cross and N.D. Morgan, Continuous Frames of Reinforced Concrete, John Wiley and Sons Inc., New York, 1932, pp. 124-125.

19. L.E. Grinter, Discussion (pp. 11-20) to H. Cross, "Analysis of Continuous Frames by Distributing Fixed-End Moments," Trans. ASCE, Vol. 96, 1932.

20. A. Pauw, "Sequence Summation Factors," Proc. ASCE, Journal of the Structural Division, Vol. 81, Aug. 1955, Paper No. 763.

21. T. Yoshimura, "Analysis of Rigid Frames by Balancing Member Series," Memoirs of the Faculty of Engineering, Kumamoto University, Vol. 2, No. 1, 1955.

22. J.J. Tuma and M.T. Anderson, "Analysis of Continuous Beams by Infinite Series," Oklahoma Engineering Experiment Station Publication, Oklahoma State University, Stillwater, No. 91, 1954.

23. J.J. Tuma, "Analysis of Continuous Beams by Carry-Over Moments," Proc. ASCE, Journal of the Structural Division, Vol. 84, Sept. 1958, Paper No. 1762.

24. E. Markland, "Properties of Circular, Parabolic and Elliptic Arches," Civil Engineering and Public Works Review, Vol. 50, No. 584, Feb. 1955, pp. 179-180.

25. A.J. Celis and J.H. Senne, "Flexibility and Stiffness Matrices for Curvilinear Structures," Proceeding of the Symposium on Shell Structures and Climatic Influences, International Association for Shell Structures, Calgary, Canada, July 1972.

26. A.J.S. Pippard and J.F. Baker, The Analysis of Engineering Structures, American Elsevier Publishing Company, Inc., New- York, Fourth Edition 1968, pp. 260-303.

27. S.T. Carpenter, Structural Mechanics, John Wiley and Sons, Inc., 1960, pp. 347-380.

28. R. Oldenburger, "Convergence of Hardy Cross's Balancing Process," Journal of Applied Mechanics, Vol. 7, (Trans. ASME, Vol. 62), 1940, pp. A166-A170.

29. C. Massonnet, "Sur la Convergence de la Méthode de Cross et l'Évaluation de l'Erreur Commise Après Chaque Étape d'Équilibrage," Revue Universelle des Mines, 9e Série, T. II, No. 6, pp. 243-247.

30. N.J. Hoff, The Analysis of Structures, John Wiley and Sons, Inc., Second Printing 1964, pp. 294-318.

31. G.E. Beggs, "An Accurate Mechanical Solution of Statically Indeterminate Structures by Use of Paper Models and Special Gages," Proc. ACI, Vol. 18, 1922, pp. 58-78.

32. G.E. Beggs, Discussion (pp. 1208-1230) to A.C. Janni, "The Design of a Multiple-Arch System and Permissible Simplifications," Trans. ASCE, Vol. 88, 1925, Paper No. 1571.

33. G.E. Beggs, "The Use of Models in the Solution of Indeterminate Structures," J. Franklin Inst., March 1927.

34. D.L. Dean, "An Economy Study of Vierendeel Trusses," Unpublished M.S. Thesis, School of Mines and Metallurgy of the University of Missouri, Rolla, 1951.

VITA

The author, Trinh-Ngoc Rang, was born on December 24, 1940 in Thua-Thien province, Viet-Nam. He received his primary and secondary education in Hue, graduated from Quoc-Hoc High School in June 1958.

He was admitted to the National Institute of Technology in 1958 and graduated in June 1962 with a degree in civil engineering.

In September 1962, he was awarded a scholarship from the Agency for International Development to pursue his graduate studies at the University of Illinois, Urbana and at Bradley University, Peoria, Ill. He received a master of science degree in civil engineering from Bradley University in May 1964.

He served as a staff menber at the National Institute of Technology Saigon, from 1964 to 1970.

In August 1970, under the sponsorship of the Agency for International Development, he entered the Graduate School of the University of Missouri at Rolla.

He is married to the former Bùi-Kim-Yến and presently they have two daughters, Trịnh-Ngọc Anh-Thủ and Trịnh-Ngọc Hương-Thủ.

His society memberships include: Chi Epsilon professional fraternity, Vietnamese Engineers' Association and American Society of Civil Engineers.

APPENDIX A

STIFFNESS, CARRY-OVER, THRUST-INDUCTION FACTORS AND FIXED-END REACTION COEFFICIENTS FOR VARIOUS TYPES OF SYMMETRICAL ARCHES

The following graphs exhibit the various structural properties for each of four types of symmetrical arches.

(1) Parabolic arch with secant variation in I with axial effect ignored, and denoted by the straight line P_s.

(2) Uniform parabolic arch

$I/AL^2 = 0$, curve P_0

$I/AL^2 = 1/20000$, curve P_1

$I/AL^2 = 1/10000$, curve P_2

$I/AL^2 = 1/5000$, curve P_3

(3) Uniform circular arch

$I/AL^2 = 0$, curve C_0

$I/AL^2 = 1/20000$, curve C_1

$I/AL^2 = 1/10000$, curve C_2

$I/AL^2 = 1/5000$, curve C_3

(4) Uniform semi-elliptical arch

$I/AL^2 = 0$, curve E_0

$I/AL^2 = 1/20000$, curve E_1

$I/AL^2 = 1/10000$, curve E_2

$I/AL^2 = 1/5000$, curve E_3

It is to be noted that the axial effect has little influence on k22 and k33 so that this effect has been ignored.

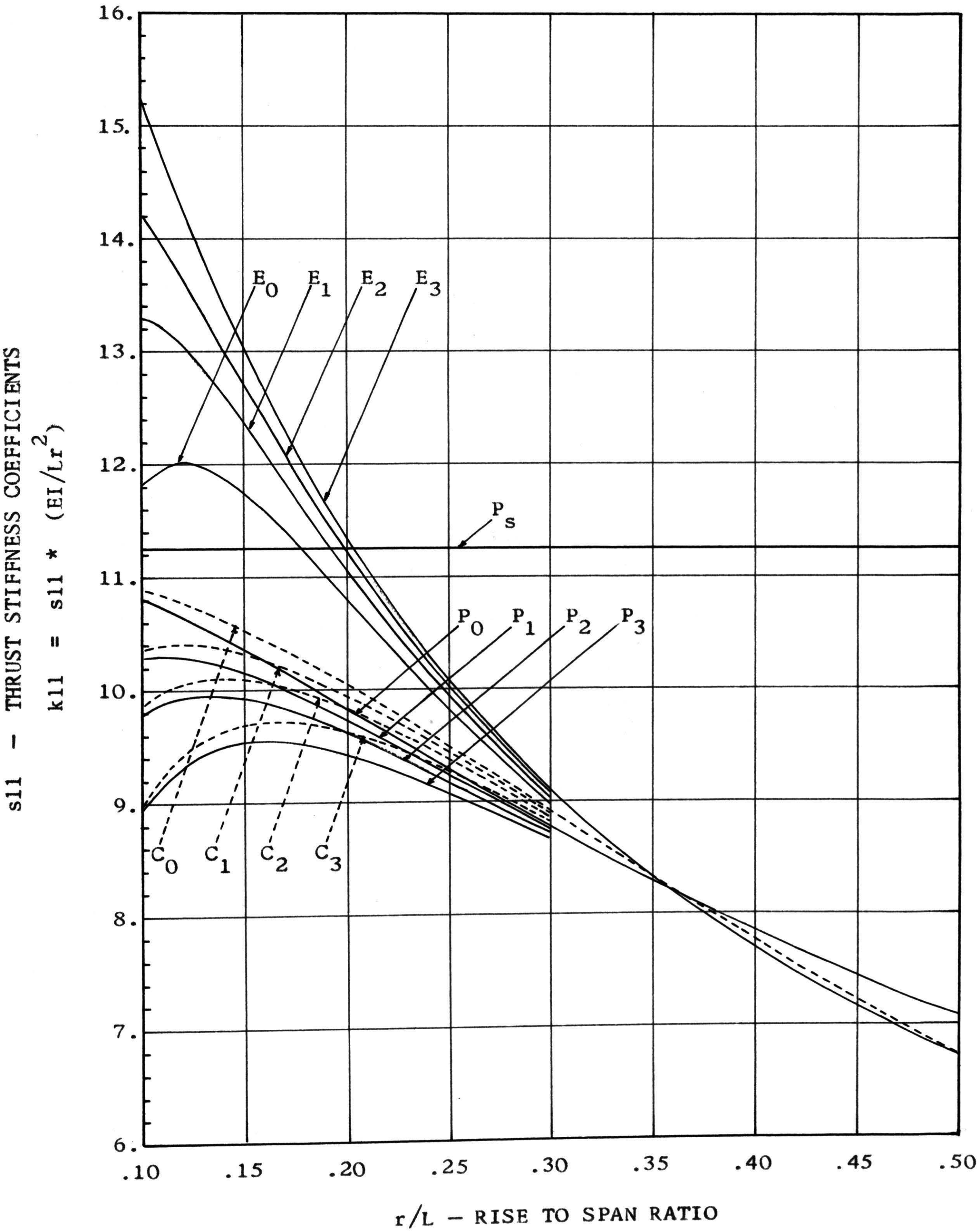


Fig. 16. Thrust Stiffness Coefficients for Various Types of Symmetrical Arches

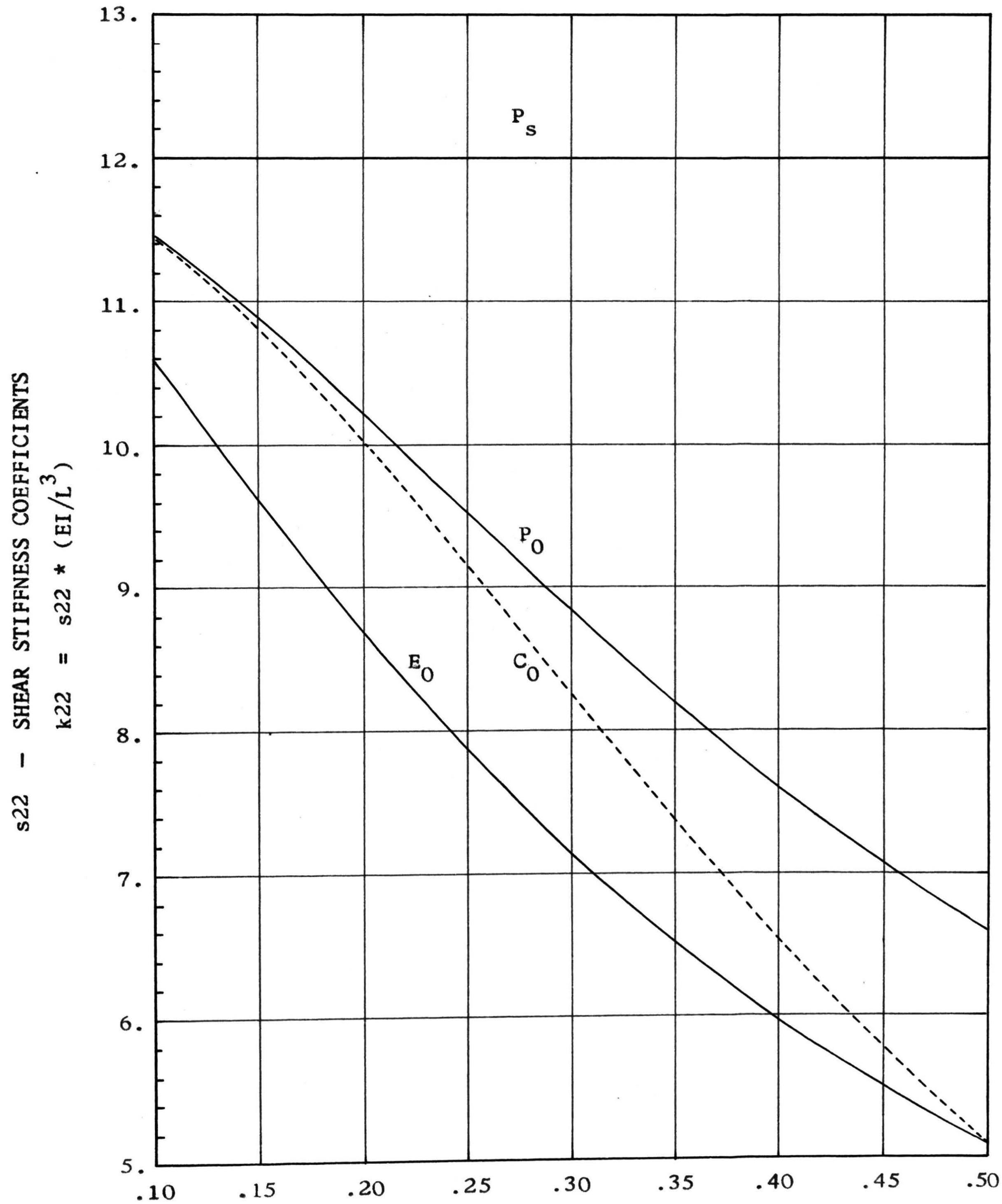


Fig. 17. Shear Stiffness Coefficients for Various Types of Symmetrical Arches

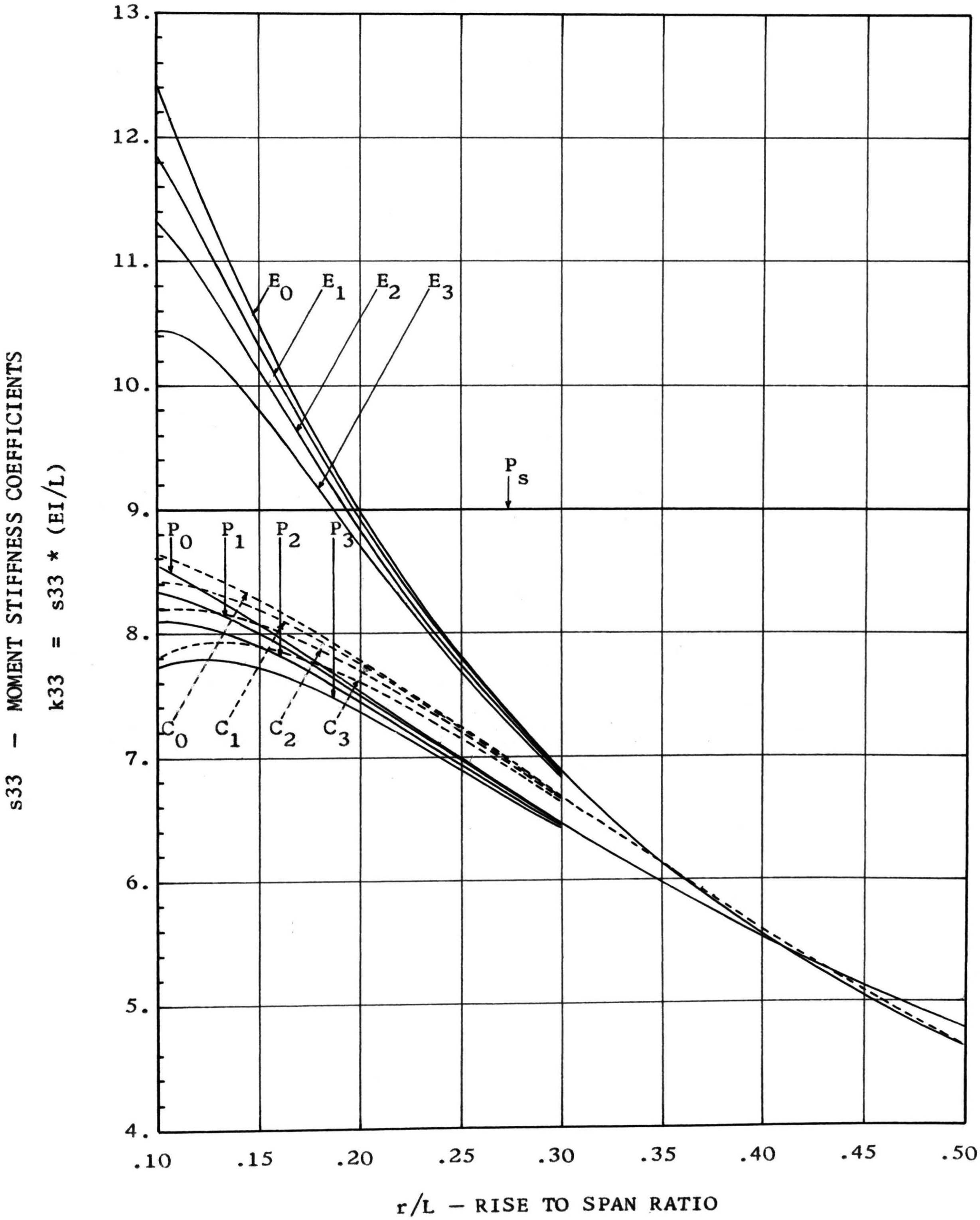


Fig. 18. Moment Stiffness Coefficients for Various Types of Symmetrical Arches

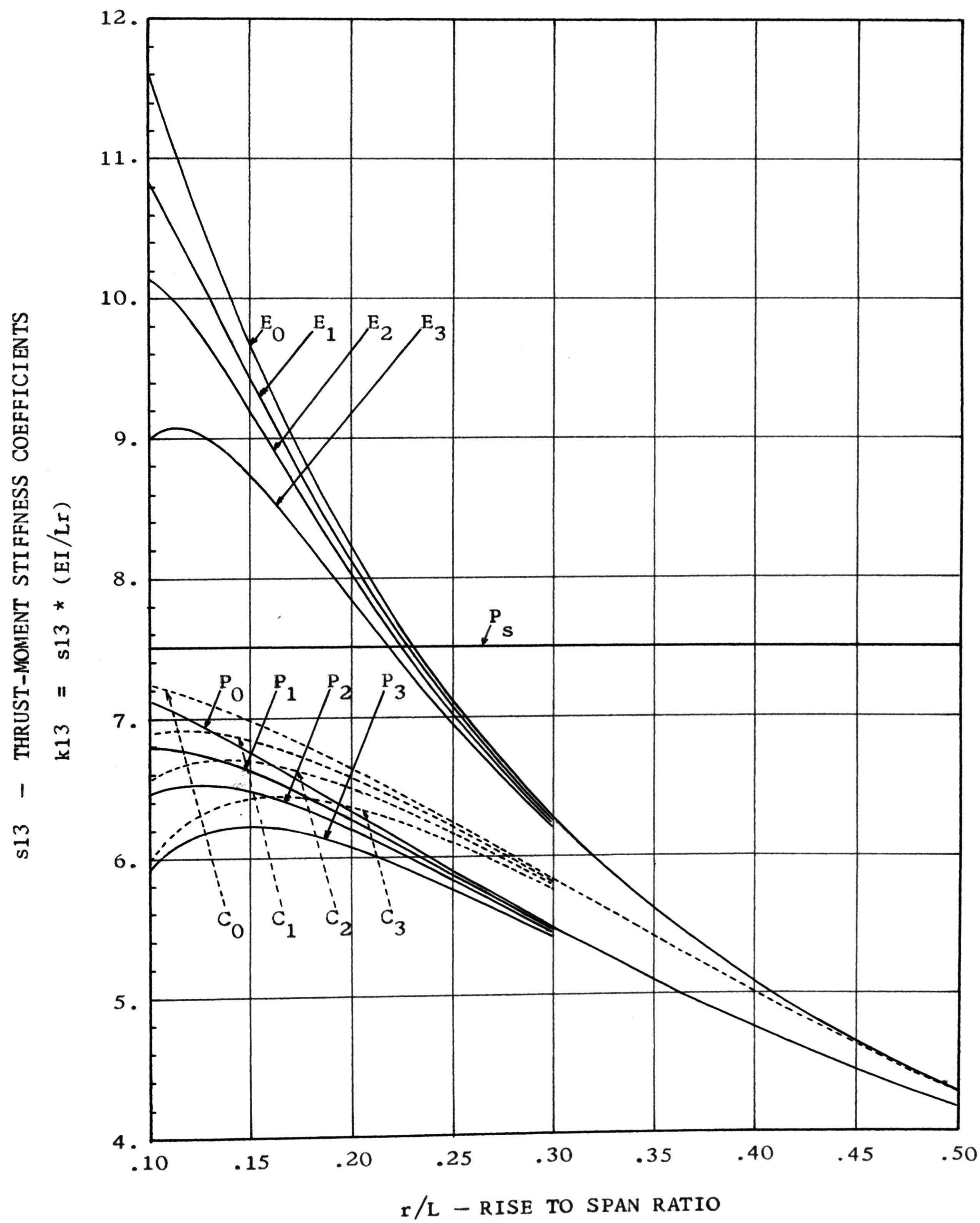


Fig. 19. Thrust-Moment Stiffness Coefficients for Various Types of Symmetrical Arches

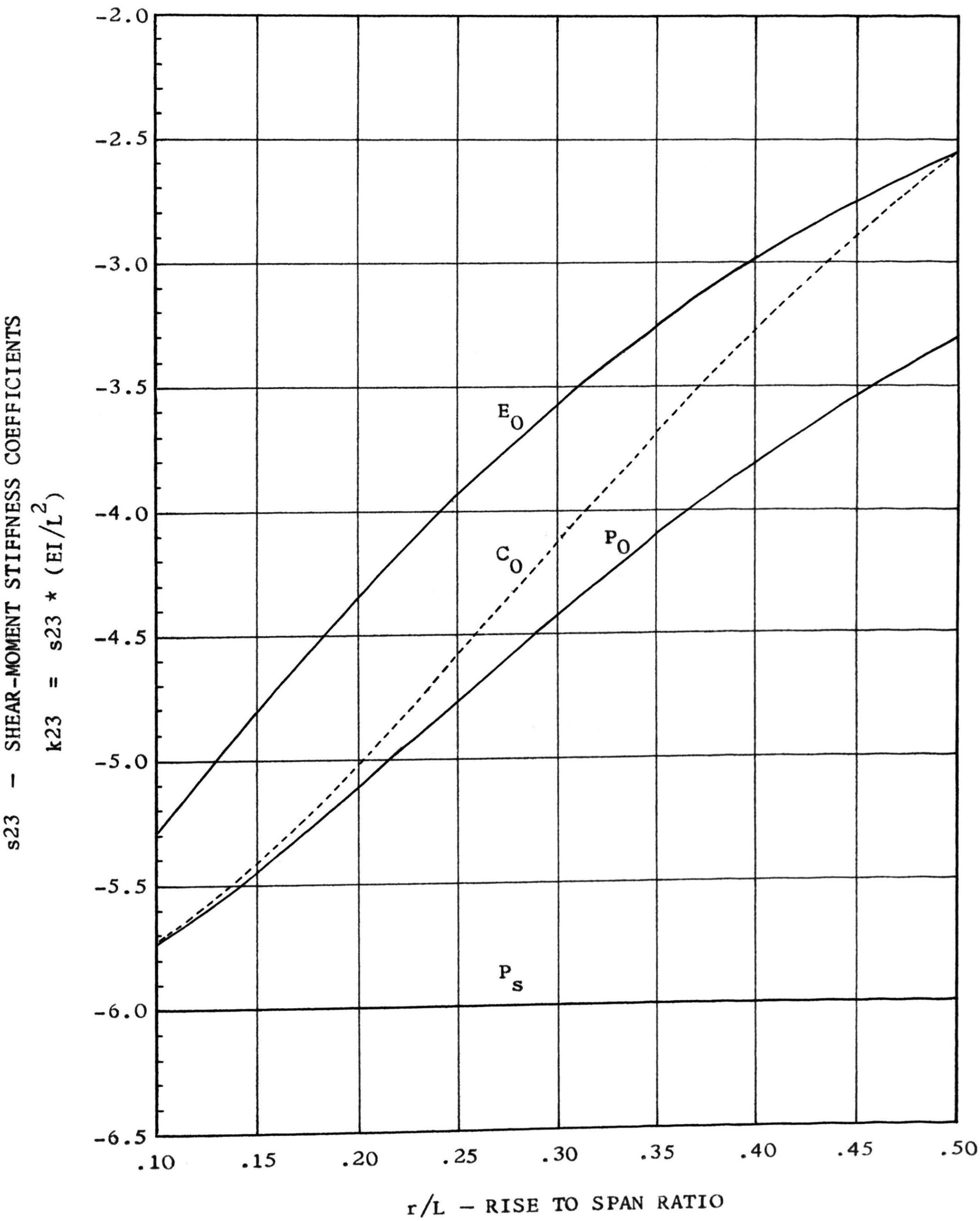


Fig. 20. Shear-Moment Stiffness Coefficients for Various Types of Symmetrical Arches

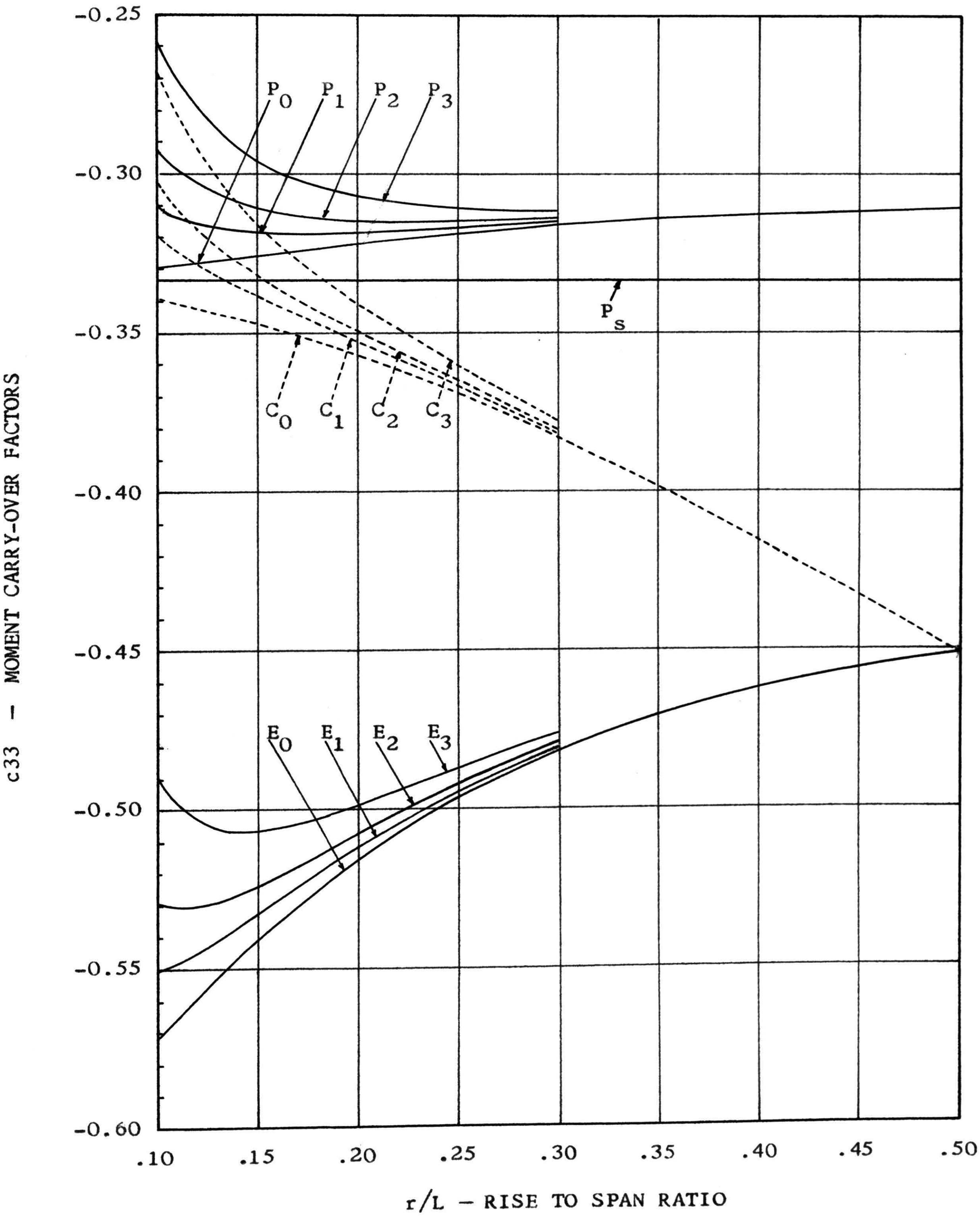


Fig. 21. Moment Carry-Over Factors for Various Types of Symmetrical Arches

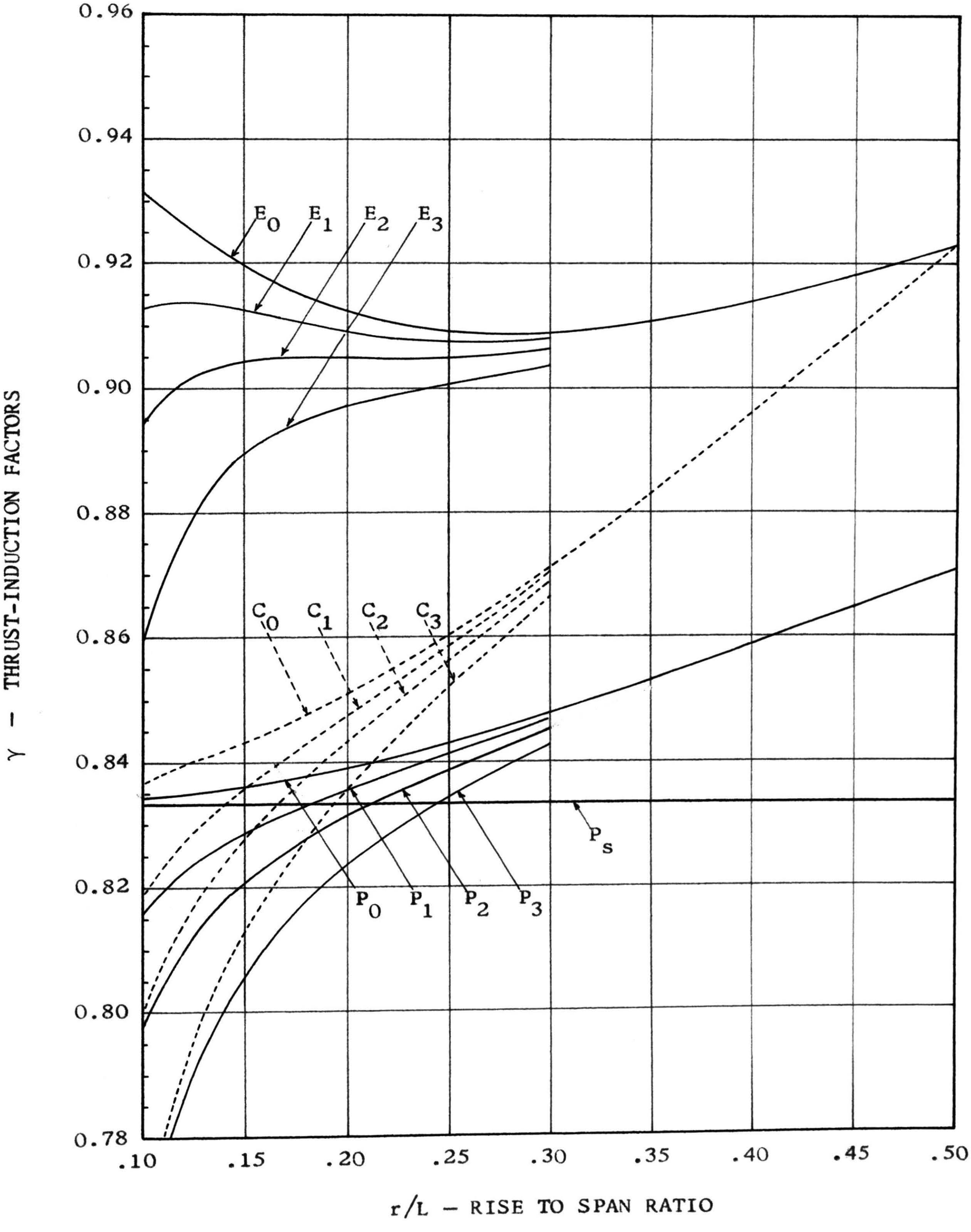


Fig. 22. Thrust-Induction Factors for Various Types of Symmetrical Arches

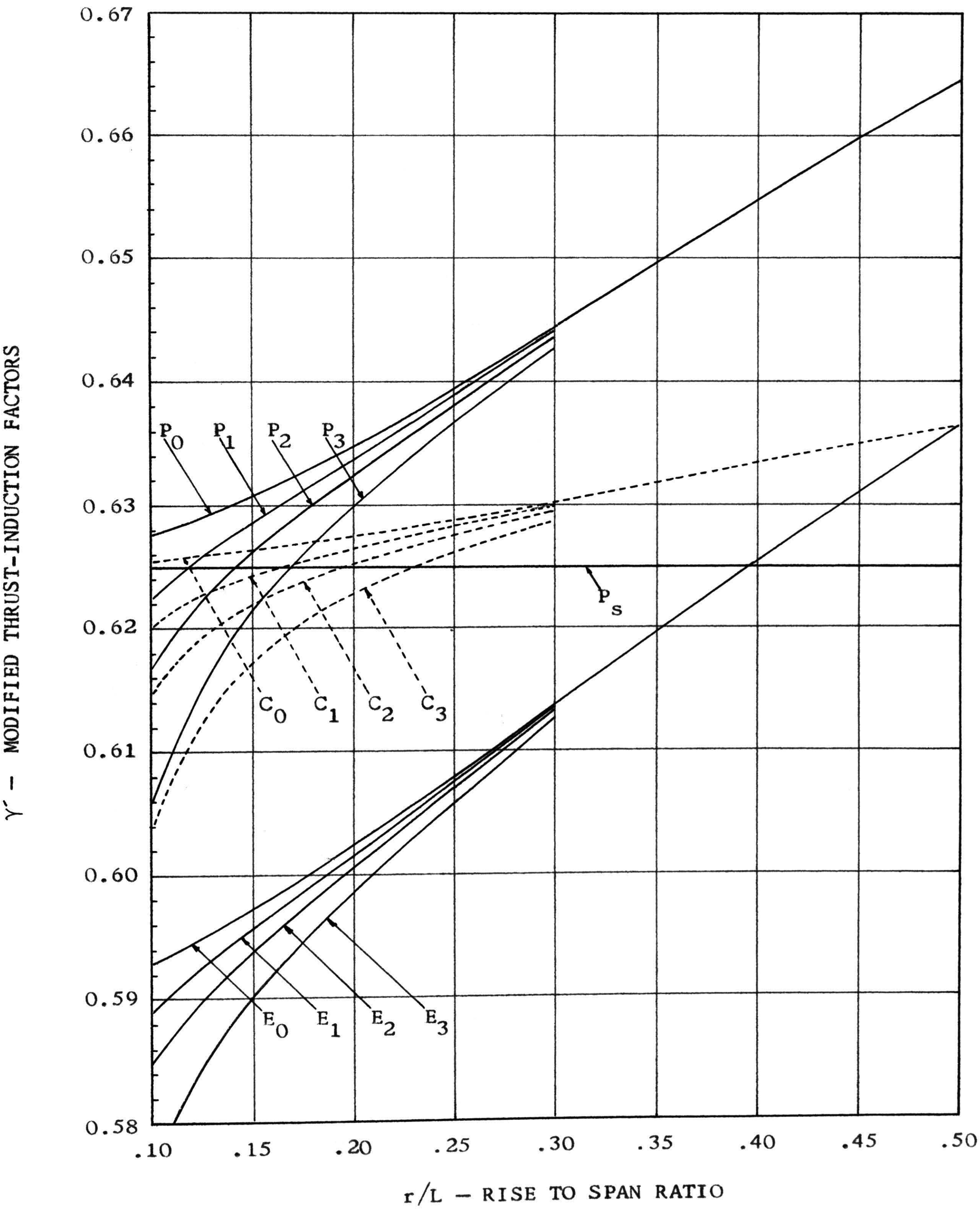


Fig. 23. Modified Thrust-Induction Factors for Various Types of Symmetrical Arches

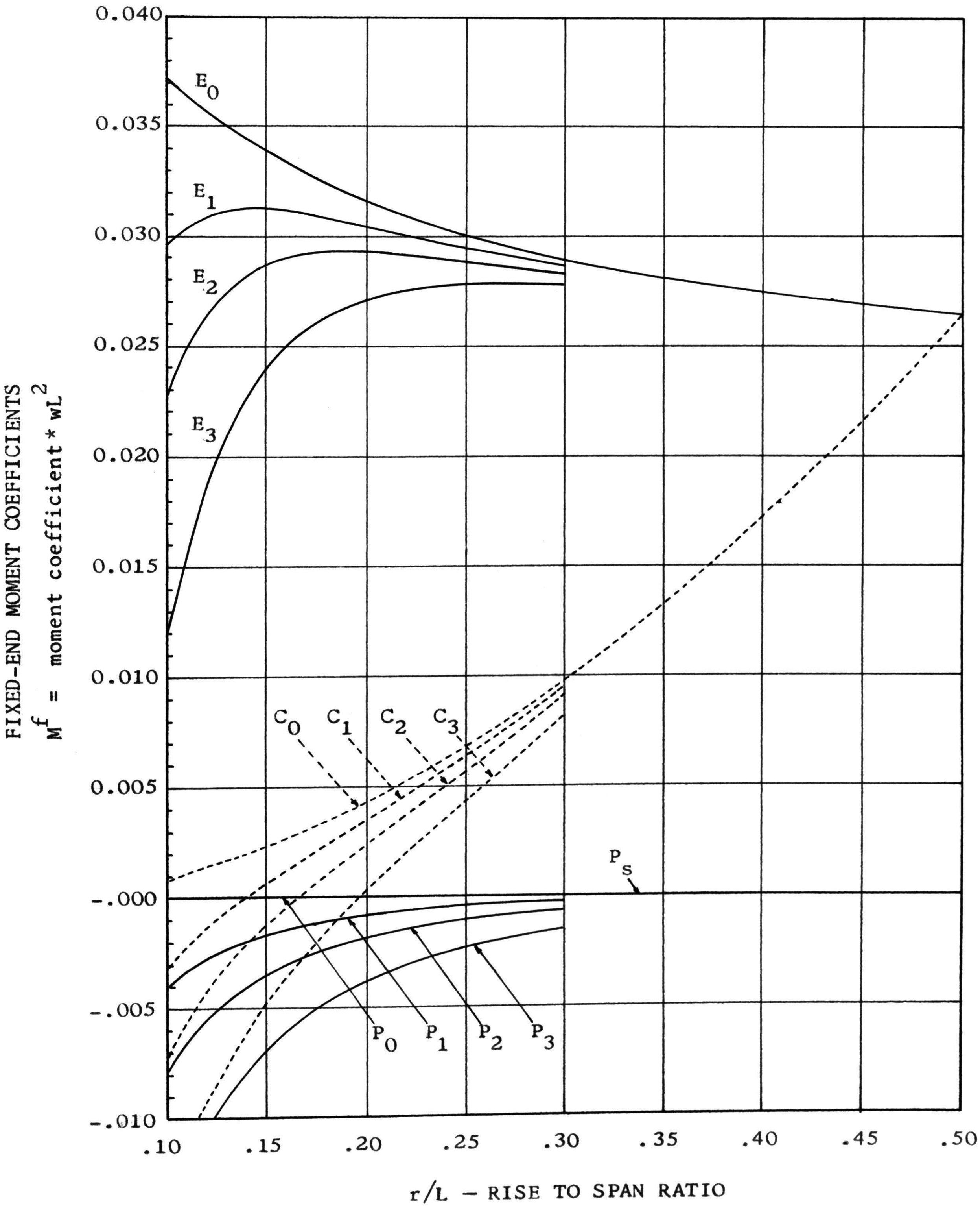


Fig. 24. Fixed-End Moment Coefficients Due to Uniformly Distributed Load for Various Types of Symmetrical Arches

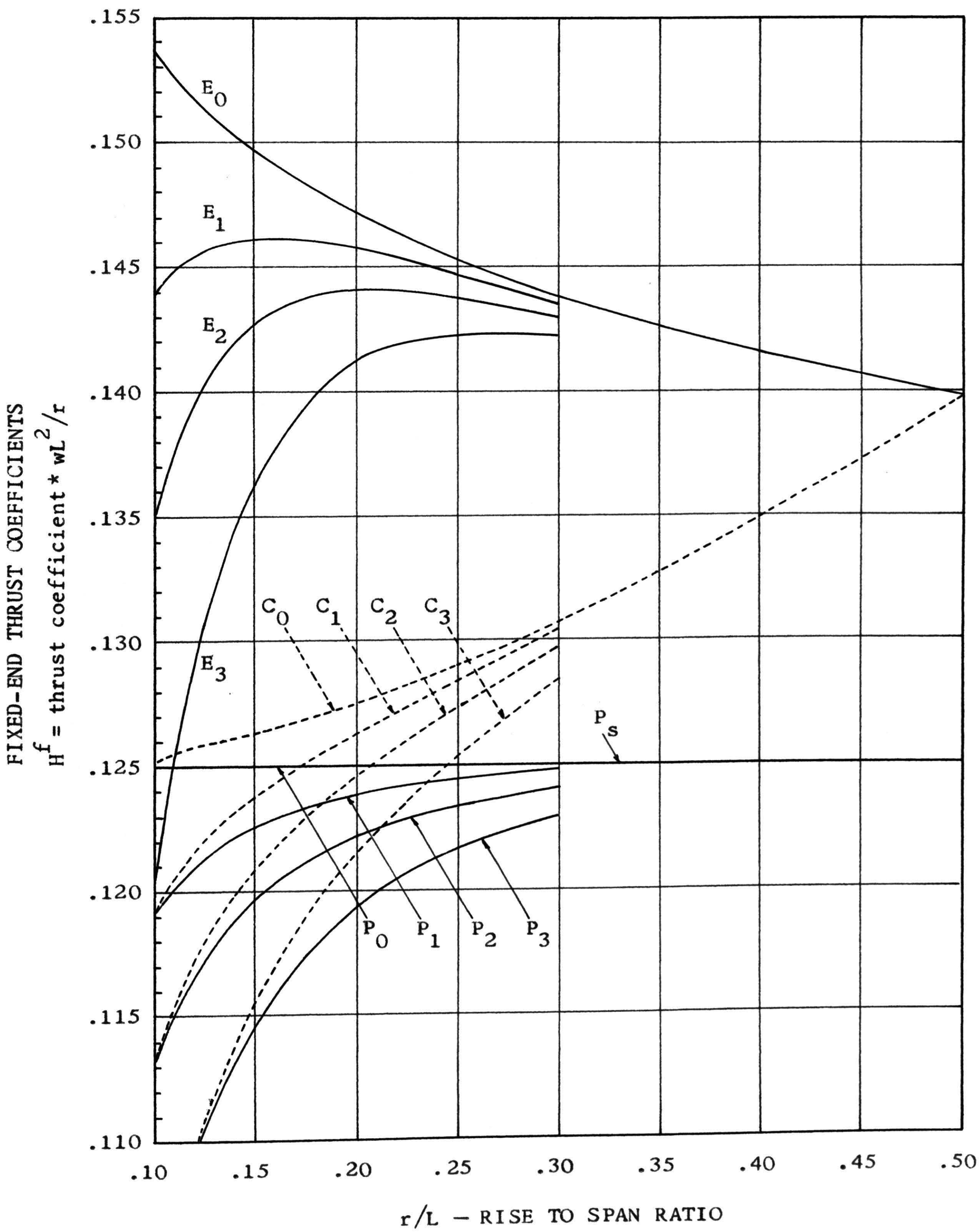


Fig. 25. Fixed-End Thrust Coefficients Due to Uniformly Distributed Load for Various Types of Symmetrical Arches

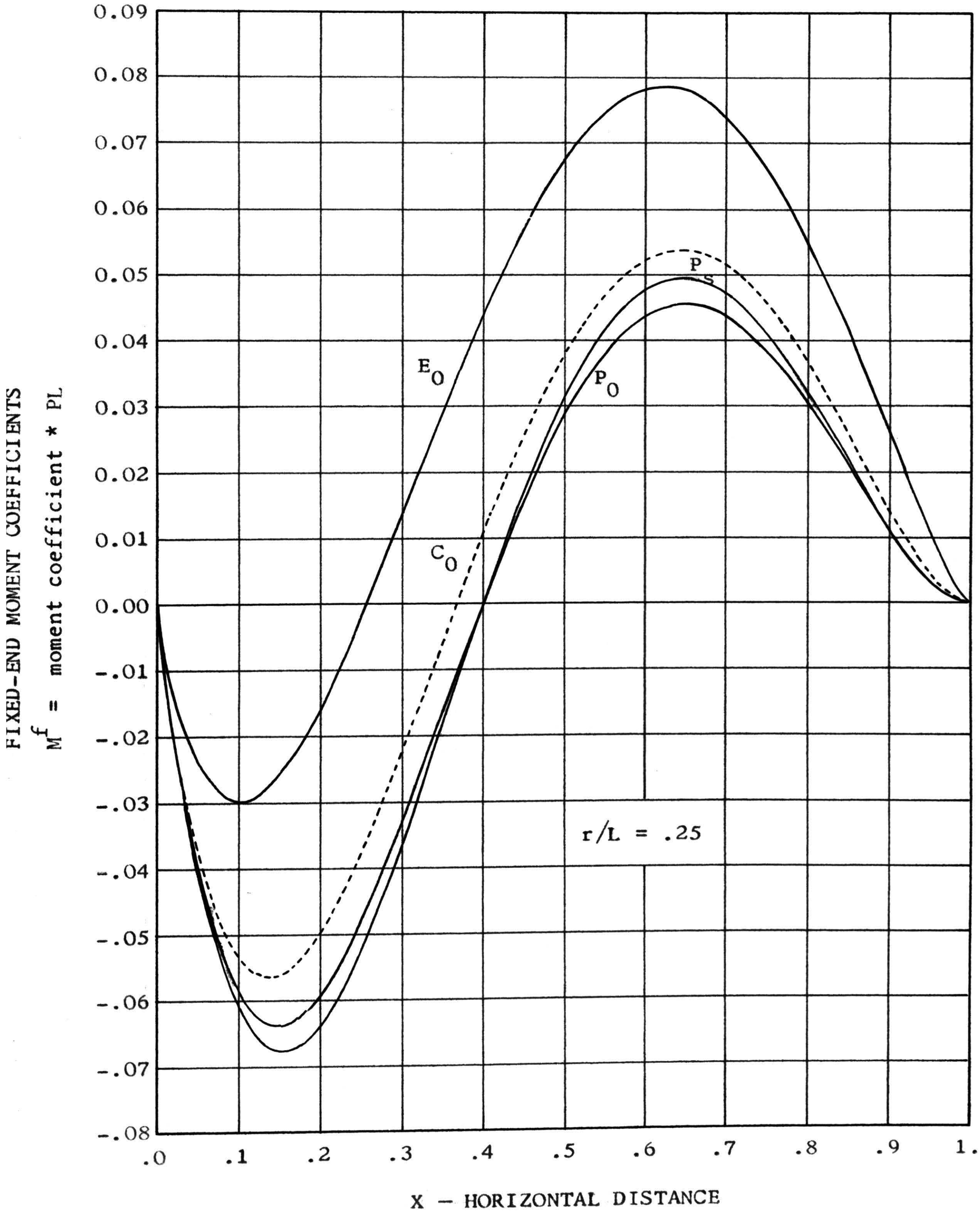


Fig. 26. Influence Line Coefficients for the Fixed-End Moments of Various Types of Symmetrical Arches

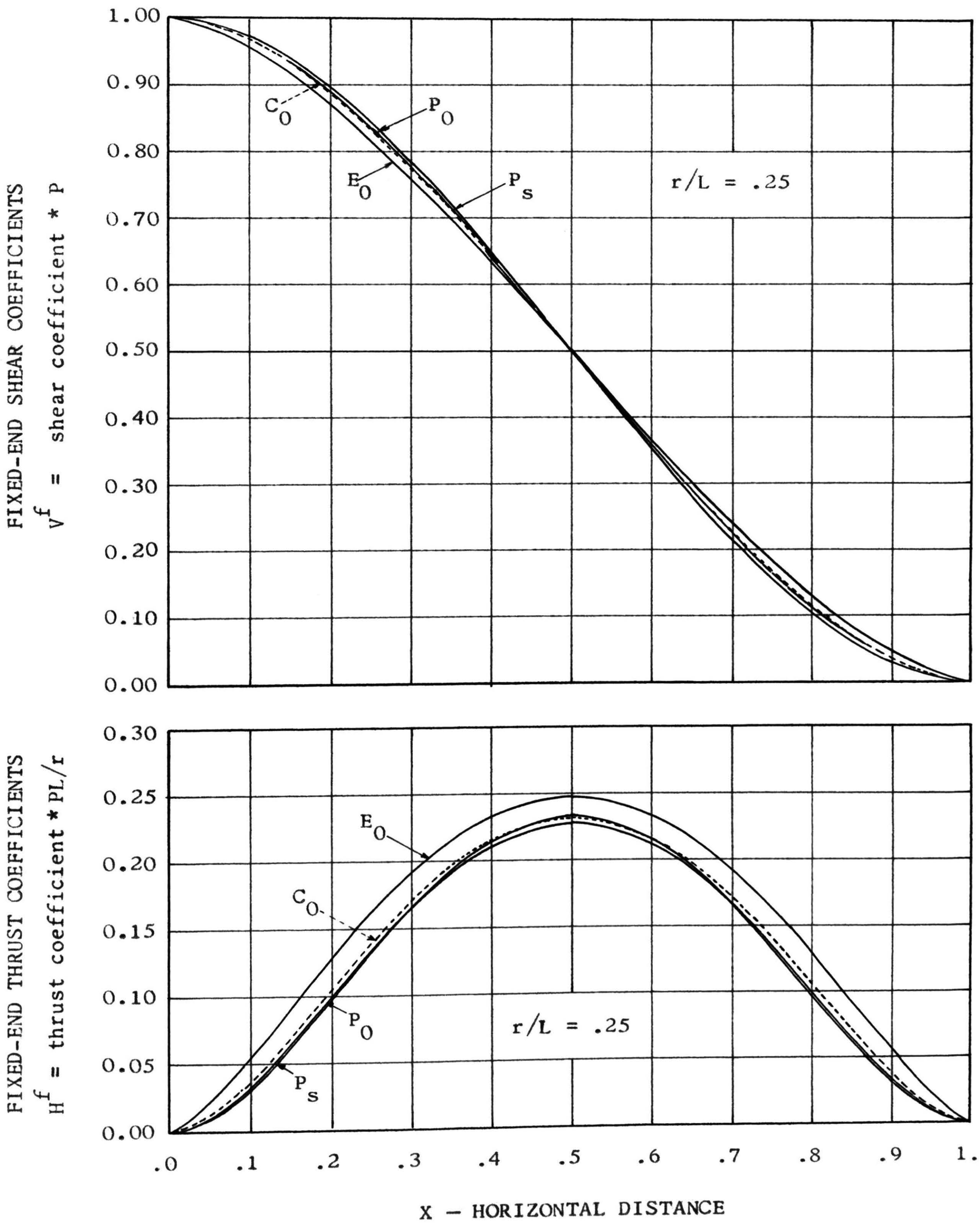


Fig. 27. Influence Line Coefficients for the Fixed-End Shears and the Fixed-End Thrusts of Various Types of Symmetrical Arches

Table VI. Influence Line Coefficients for the Fixed-End Reactions at the Left-Hand End of Symmetrical Parabolic Arches with Secant Variation in I ($I/AL^2 = 0$)

M^f = moment coefficient * PL/10

.1	.2	.3	.4	.5	.6	.7	.8	.9
-.6075	-.6400	-.3675	.0000	.3125	.4800	.4725	.3200	.1125

H^f = thrust coefficient * PL/r

.1	.2	.3	.4	.5	.6	.7	.8	.9
.0304	.0960	.1654	.2160	.2344	.2160	.1654	.0960	.0304

V^f = shear coefficient * P

.1	.2	.3	.4	.5	.6	.7	.8	.9
.9720	.8960	.7840	.6480	.5000	.3520	.2160	.1040	.0280

Table VII. Influence Line Coefficients for the Fixed-End Moment at the Left-Hand End of Uniform Circular Arches with Various Values of I/AL^2

$$M^f = \text{moment coefficient} * PL/10$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	-.5966	-.6172	-.3432	.0180	.3232	.4871	.4799	.3280	.1170
.15	-.5824	-.5886	-.3134	.0400	.3363	.4958	.4889	.3378	.1228
.20	-.5616	-.5486	-.2727	.0700	.3545	.5077	.5009	.3510	.1312
.25	-.5332	-.4974	-.2220	.1073	.3775	.5228	.5156	.3674	.1422
.30	-.4959	-.4354	-.1625	.1510	.4049	.5408	.5325	.3864	.1562
.35	-.4486	-.3633	-.0953	.2004	.4364	.5616	.5513	.4075	.1733
.40	-.3901	-.2820	-.0217	.2547	.4719	.5852	.5717	.4303	.1934
.45	-.3198	-.1928	.0572	.3133	.5108	.6113	.5937	.4543	.2163
.50	-.2379	-.0970	.1404	.3753	.5530	.6400	.6172	.4792	.2417
				$I/AL^2 = 1/20000$					
.10	-.6083	-.6520	-.4019	-.0578	.2412	.4112	.4212	.2931	.1053
.15	-.5882	-.6048	-.3401	.0059	.2996	.4616	.4621	.3214	.1170
.20	-.5651	-.5579	-.2876	.0511	.3342	.4887	.4858	.3414	.1275
.25	-.5355	-.5034	-.2314	.0955	.3648	.5108	.5059	.3610	.1396
.30	-.4976	-.4395	-.1688	.1431	.3964	.5327	.5257	.3818	.1542
				$I/AL^2 = 1/10000$					
.10	-.6188	-.6835	-.4549	-.1263	.1671	.3427	.3681	.2615	.0947
.15	-.5936	-.6203	-.3656	-.0267	.2644	.4289	.4364	.3057	.1114
.20	-.5685	-.5670	-.3022	.0327	.3143	.4701	.4709	.3321	.1239
.25	-.5378	-.5093	-.2407	.0839	.3523	.4991	.4963	.3548	.1370
.30	-.4992	-.4436	-.1751	.1353	.3880	.5247	.5191	.3773	.1522
				$I/AL^2 = 1/5000$					
.10	-.6371	-.7382	-.5470	-.2452	.0384	.2237	.2759	.2066	.0763
.15	-.6039	-.6495	-.4135	-.0879	.1983	.3675	.3882	.2762	.1008
.20	-.5750	-.5846	-.3305	-.0030	.2759	.4342	.4422	.3139	.1169
.25	-.5423	-.5209	-.2589	.0612	.3279	.4760	.4775	.3424	.1319
.30	-.5024	-.4516	-.1875	.1199	.3714	.5088	.5059	.3683	.1482

Table VIII. Influence Line Coefficients for the Fixed-End Thrust at the Left-Hand End of Uniform Circular Arches with Various Values of I/AL^2

$$H^f = \text{thrust coefficient} * PL/r$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	.0313	.0975	.1664	.2162	.2341	.2162	.1664	.0975	.0313
.15	.0325	.0994	.1677	.2163	.2338	.2163	.1677	.0994	.0325
.20	.0341	.1020	.1693	.2165	.2334	.2165	.1693	.1020	.0341
.25	.0364	.1052	.1712	.2168	.2329	.2168	.1712	.1052	.0364
.30	.0393	.1089	.1733	.2170	.2323	.2170	.1733	.1089	.0393
.35	.0429	.1129	.1755	.2172	.2316	.2172	.1755	.1129	.0429
.40	.0472	.1173	.1777	.2173	.2310	.2173	.1777	.1173	.0472
.45	.0521	.1217	.1799	.2174	.2303	.2174	.1799	.1217	.0521
.50	.0575	.1262	.1819	.2174	.2296	.2174	.1819	.1262	.0575
				$I/AL^2 = 1/20000$					
.10	.0295	.0923	.1576	.2048	.2218	.2048	.1576	.0923	.0295
.15	.0316	.0970	.1636	.2112	.2283	.2112	.1636	.0970	.0316
.20	.0336	.1006	.1670	.2137	.2303	.2137	.1670	.1006	.0336
.25	.0360	.1042	.1698	.2150	.2309	.2150	.1698	.1042	.0360
.30	.0390	.1082	.1723	.2158	.2310	.2158	.1723	.1082	.0390
				$I/AL^2 = 1/10000$					
.10	.0279	.0875	.1496	.1945	.2106	.1945	.1496	.0875	.0279
.15	.0307	.0946	.1598	.2063	.2229	.2063	.1598	.0946	.0307
.20	.0331	.0992	.1648	.2109	.2273	.2109	.1648	.0992	.0331
.25	.0357	.1033	.1683	.2132	.2290	.2132	.1683	.1033	.0357
.30	.0388	.1075	.1714	.2145	.2297	.2145	.1714	.1075	.0388
				$I/AL^2 = 1/5000$					
.10	.0252	.0793	.1358	.1766	.1913	.1766	.1358	.0793	.0252
.15	.0292	.0902	.1525	.1970	.2130	.1970	.1525	.0902	.0292
.20	.0321	.0965	.1605	.2054	.2215	.2054	.1605	.0965	.0321
.25	.0349	.1015	.1655	.2097	.2253	.2097	.1655	.1015	.0349
.30	.0382	.1062	.1694	.2122	.2271	.2122	.1694	.1062	.0382

Table IX. Influence Line Coefficients for the Fixed-End Shear at the Left-Hand End of Uniform Circular Arches with Various Values of I/AL^2

$$V^f = \text{shear coefficient} * P$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	.9714	.8945	.7823	.6469	.5000	.3531	.2177	.1055	.0286
.15	.9705	.8926	.7802	.6456	.5000	.3544	.2198	.1074	.0295
.20	.9693	.8900	.7774	.6438	.5000	.3562	.2226	.1100	.0307
.25	.9675	.8865	.7738	.6415	.5000	.3585	.2262	.1135	.0325
.30	.9652	.8822	.7695	.6390	.5000	.3610	.2305	.1178	.0348
.35	.9622	.8771	.7647	.6361	.5000	.3639	.2353	.1229	.0378
.40	.9583	.8712	.7593	.6330	.5000	.3670	.2407	.1288	.0417
.45	.9536	.8647	.7536	.6298	.5000	.3702	.2464	.1353	.0464
.50	.9480	.8576	.7477	.6265	.5000	.3735	.2523	.1424	.0520
				$I/AL^2 = 1/20000$					
.10	.9714	.8945	.7823	.6469	.5000	.3531	.2117	.1055	.0286
.15	.9705	.8926	.7802	.6456	.5000	.3544	.2198	.1074	.0295
.20	.9693	.8899	.7773	.6438	.5000	.3562	.2227	.1101	.0307
.25	.9675	.8864	.7737	.6415	.5000	.3585	.2263	.1136	.0325
.30	.9652	.8821	.7695	.6390	.5000	.3610	.2305	.1179	.0348
				$I/AL^2 = 1/10000$					
.10	.9714	.8945	.7823	.6469	.5000	.3531	.2177	.1055	.0286
.15	.9705	.8926	.7802	.6456	.5000	.3544	.2198	.1074	.0295
.20	.9692	.8899	.7773	.6437	.5000	.3563	.2227	.1101	.0308
.25	.9675	.8864	.7737	.6415	.5000	.3585	.2263	.1136	.0325
.30	.9651	.8821	.7694	.6389	.5000	.3611	.2306	.1179	.0349
				$I/AL^2 = 1/5000$					
.10	.9713	.8945	.7823	.6469	.5000	.3531	.2177	.1055	.0287
.15	.9705	.8926	.7802	.6455	.5000	.3545	.2198	.1074	.0295
.20	.9692	.8899	.7773	.6437	.5000	.3563	.2227	.1101	.0308
.25	.9674	.8863	.7736	.6415	.5000	.3585	.2264	.1137	.0326
.30	.9651	.8820	.7693	.6389	.5000	.3611	.2307	.1180	.0349

Table X. Influence Line Coefficients for the Fixed-End Moment at the Left-Hand End of Uniform Parabolic Arches with Various Values of I/AL^2

$$M^f = \text{moment coefficient} * PL/10$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	-.6018	-.6292	-.3592	.0007	.3060	.4706	.4653	.3173	.1126
.15	-.5959	-.6180	-.3504	.0016	.2992	.4608	.4578	.3145	.1127
.20	-.5894	-.6054	-.3405	.0026	.2914	.4496	.4493	.3114	.1128
.25	-.5829	-.5928	-.3304	.0037	.2836	.4384	.4409	.3084	.1129
.30	-.5769	-.5809	-.3208	.0047	.2761	.4277	.4331	.3057	.1131
.35	-.5715	-.5702	-.3122	.0055	.2692	.4180	.4261	.3035	.1133
.40	-.5667	-.5607	-.3045	.0062	.2628	.4093	.4201	.3018	.1136
.45	-.5626	-.5524	-.2978	.0065	.2570	.4015	.4150	.3004	.1140
.50	-.5590	-.5451	-.2919	.0068	.2517	.3947	.4106	.2995	.1144
				$I/AL^2 = 1/20000$					
.10	-.6131	-.6632	-.4166	-.0736	.2255	.3962	.4078	.2833	.1012
.15	-.6012	-.6334	-.3759	-.0312	.2637	.4279	.4321	.2990	.1072
.20	-.5925	-.6140	-.3545	-.0153	.2721	.4316	.4351	.3026	.1095
.25	-.5849	-.5982	-.3391	-.0073	.2717	.4272	.4319	.3027	.1106
.30	-.5782	-.5845	-.3266	-.0027	.2681	.4201	.4269	.3017	.1114
				$I/AL^2 = 1/10000$					
.10	-.6233	-.6939	-.4685	-.1408	.1528	.3290	.3558	.2525	.0909
.15	-.6063	-.6481	-.4004	-.0626	.2298	.3964	.4075	.2841	.1020
.20	-.5955	-.6224	-.3682	-.0327	.2533	.4140	.4212	.2940	.1063
.25	-.5869	-.6435	-.3476	-.0182	.2599	.4161	.4231	.2971	.1084
.30	-.5796	-.5881	-.3324	-.0100	.2602	.4126	.4208	.2977	.1098
				$I/AL^2 = 1/5000$					
.10	-.6411	-.7473	-.5587	-.2576	.0263	.2121	.2655	.1989	.0730
.15	-.6160	-.6758	-.4464	-.1217	.1660	.3371	.3612	.2561	.0921
.20	-.6014	-.6386	-.3947	-.0666	.2167	.3799	.3942	.2772	.1000
.25	-.5907	-.6138	-.3644	-.0395	.2369	.3944	.4057	.2860	.1041
.30	-.5822	-.5952	-.3437	-.0244	.2446	.3978	.4087	.2899	.1065

Table XI. Influence Line Coefficients for the Fixed-End Thrust at the Left-Hand End of Uniform Parabolic Arches with Various Values of I/AL^2

$$H^f = \text{thrust coefficient} * PL/r$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	.0308	.0966	.1655	.2153	.2334	.2153	.1655	.0966	.0308
.15	.0313	.0973	.1656	.2146	.2323	.2146	.1656	.0973	.0313
.20	.0319	.0981	.1657	.2138	.2311	.2138	.1657	.0981	.0319
.25	.0324	.0989	.1659	.2129	.2297	.2129	.1659	.0989	.0324
.30	.0330	.0997	.1660	.2121	.2283	.2121	.1660	.0997	.0330
.35	.0335	.1006	.1662	.2112	.2270	.2112	.1662	.1006	.0335
.40	.0340	.1013	.1663	.2104	.2257	.2104	.1663	.1013	.0340
.45	.0345	.1021	.1665	.2096	.2245	.2096	.1665	.1021	.0345
.50	.0350	.1028	.1667	.2089	.2233	.2089	.1667	.1028	.0350
				$I/AL^2 = 1/20000$					
.10	.0291	.0915	.1568	.2041	.2212	.2041	.1568	.0915	.0291
.15	.0305	.0949	.1616	.2096	.2269	.2096	.1616	.0949	.0305
.20	.0314	.0967	.1635	.2110	.2281	.2110	.1635	.0967	.0314
.25	.0321	.0981	.1645	.2112	.2278	.2112	.1645	.0981	.0321
.30	.0328	.0991	.1651	.2109	.2271	.2109	.1651	.0991	.0328
				$I/AL^2 = 1/10000$					
.10	.0276	.0868	.1489	.1939	.2102	.1939	.1489	.0868	.0276
.15	.0297	.0926	.1579	.2048	.2217	.2048	.1579	.0926	.0297
.20	.0309	.0954	.1614	.2083	.2251	.2083	.1614	.0954	.0309
.25	.0318	.0972	.1631	.2095	.2260	.2095	.1631	.0972	.0318
.30	.0325	.0985	.1641	.2097	.2258	.2097	.1641	.0985	.0325
				$I/AL^2 = 1/5000$					
.10	.0249	.0787	.1352	.1762	.1910	.1762	.1352	.0787	.0249
.15	.0282	.0884	.1508	.1957	.2119	.1957	.1508	.0884	.0282
.20	.0300	.0929	.1572	.2030	.2194	.2030	.1572	.0929	.0300
.25	.0311	.0955	.1604	.2061	.2223	.2061	.1604	.0955	.0311
.30	.0320	.0973	.1622	.2073	.2233	.2073	.1622	.0973	.0320

Table XII. Influence Line Coefficients for the Fixed-End Shear at the Left-Hand End of Uniform Parabolic Arches with Various Values of I/AL^2

$$V^f = \text{shear coefficient} * P$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	.9714	.8946	.7824	.6470	.5000	.3530	.2176	.1054	.0286
.15	.9708	.8932	.7808	.6459	.5000	.3541	.2192	.1068	.0292
.20	.9702	.8916	.7789	.6447	.5000	.3553	.2211	.1084	.0298
.25	.9696	.8901	.7771	.6435	.5000	.3565	.2229	.1099	.0304
.30	.9690	.8887	.7754	.6423	.5000	.3577	.2246	.1113	.0310
.35	.9685	.8874	.7738	.6412	.5000	.3588	.2262	.1126	.0315
.40	.9680	.8862	.7725	.6403	.5000	.3597	.2275	.1138	.0320
.45	.9676	.8853	.7713	.6395	.5000	.3605	.2287	.1147	.0324
.50	.9673	.8844	.7702	.6388	.5000	.3612	.2298	.1156	.0327
				$I/AL^2 = 1/20000$					
.10	.9714	.8946	.7824	.6470	.5000	.3530	.2176	.1054	.0286
.15	.9708	.8932	.7808	.6459	.5000	.3541	.2192	.1068	.0292
.20	.9702	.8916	.7789	.6447	.5000	.3553	.2211	.1084	.0298
.25	.9695	.8901	.7771	.6434	.5000	.3566	.2229	.1099	.0305
.30	.9689	.8886	.7753	.6423	.5000	.3577	.2247	.1114	.0311
				$I/AL^2 = 1/10000$					
.10	.9714	.8946	.7824	.6470	.5000	.3530	.2176	.1054	.0286
.15	.9708	.8932	.7808	.6459	.5000	.3541	.2192	.1068	.0292
.20	.9702	.8916	.7789	.6447	.5000	.3553	.2211	.1084	.0298
.25	.9695	.8900	.7771	.6434	.5000	.3566	.2229	.1100	.0305
.30	.9689	.8886	.7753	.6423	.5000	.3577	.2247	.1114	.0311
				$I/AL^2 = 1/5000$					
.10	.9714	.8946	.7824	.6470	.5000	.3530	.2176	.1054	.0286
.15	.9708	.8932	.7808	.6459	.5000	.3541	.2192	.1068	.0292
.20	.9701	.8916	.7789	.6446	.5000	.3554	.2211	.1084	.0299
.25	.9695	.8900	.7770	.6434	.5000	.3566	.2230	.1100	.0305
.30	.9688	.8885	.7752	.6422	.5000	.3578	.2248	.1115	.0312

Table XIII. Influence Line Coefficients for the Fixed-End Moment at the Left-Hand End of Uniform Semi-Elliptical Arches with Various Values of I/AL^2

$$M^f = \text{moment coefficient} * PL/10$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	-.3872	-.2274	.1520	.5577	.8621	.9886	.9077	.6402	.2702
.15	-.3515	-.2047	.1344	.4974	.7737	.8958	.8348	.6034	.2675
.20	-.3226	-.1820	.1277	.4577	.7107	.8266	.7783	.5730	.2630
.25	-.2997	-.1615	.1271	.4313	.6650	.7747	.7345	.5483	.2582
.30	-.2815	-.1439	.1290	.4131	.6311	.7349	.7002	.5283	.2537
.35	-.2671	-.1290	.1319	.4000	.6050	.7036	.6729	.5121	.2498
.40	-.2554	-.1164	.1350	.3899	.5843	.6783	.6508	.4989	.2465
.45	-.2458	-.1059	.1378	.3819	.5673	.6575	.6325	.4881	.2438
.50	-.2379	-.0970	.1404	.3753	.5530	.6400	.6172	.4792	.2417
				$I/AL^2 = 1/20000$					
.10	-.4142	-.2973	.0423	.4207	.7156	.8519	.7983	.5707	.2434
.15	-.3625	-.2320	.0924	.4456	.7185	.8441	.7930	.5762	.2564
.20	-.3282	-.1955	.1075	.4328	.6841	.8017	.7579	.5594	.2572
.25	-.3029	-.1692	.1157	.4174	.6502	.7606	.7229	.5403	.2546
.30	-.2836	-.1487	.1219	.4045	.6219	.7260	.6928	.5231	.2512
				$I/AL^2 = 1/10000$					
.10	-.4377	-.3581	-.0533	.3014	.5880	.7327	.7031	.5101	.2200
.15	-.3730	-.2579	.0527	.3965	.6662	.7951	.7534	.5504	.2459
.20	-.3336	-.2085	.0877	.4085	.6583	.7774	.7380	.5462	.2515
.25	-.3061	-.1767	.1044	.4037	.6356	.7468	.7115	.5325	.2510
.30	-.2856	-.1534	.1148	.3959	.6127	.7173	.6855	.5180	.2487
				$I/AL^2 = 1/5000$					
.10	-.4766	-.4590	-.2117	.1036	.3765	.5353	.5452	.4097	.1813
.15	-.3923	-.3058	-.0209	.3056	.5693	.7044	.6800	.5026	.2264
.20	-.3441	-.2337	.0497	.3619	.6086	.7307	.6998	.5207	.2405
.25	-.3123	-.1914	.0824	.3768	.6070	.7198	.6891	.5181	.2441
.30	-.2895	-.1628	.1009	.3789	.5946	.7000	.6710	.5078	.2439

Table XIV. Influence Line Coefficients for the Fixed-End Thrust at the Left-Hand End of Uniform Semi-Elliptical Arches with Various Values of I/AL^2

$$H^f = \text{thrust coefficient} * PL/r$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	.0495	.1283	.2021	.2528	.2708	.2528	.2021	.1283	.0495
.15	.0515	.1274	.1967	.2437	.2602	.2437	.1967	.1274	.0515
.20	.0530	.1270	.1928	.2370	.2525	.2370	.1928	.1270	.0530
.25	.0541	.1267	.1899	.2320	.2466	.2320	.1899	.1267	.0541
.30	.0551	.1265	.1877	.2280	.2420	.2280	.1877	.1265	.0551
.35	.0558	.1264	.1859	.2247	.2382	.2247	.1859	.1264	.0558
.40	.0565	.1263	.1844	.2220	.2349	.2220	.1844	.1263	.0565
.45	.0571	.1262	.1831	.2196	.2321	.2196	.1831	.1262	.0571
.50	.0575	.1262	.1819	.2174	.2296	.2174	.1819	.1262	.0575
				$I/AL^2 = 1/20000$					
.10	.0457	.1187	.1872	.2342	.2509	.2342	.1872	.1187	.0457
.15	.0498	.1235	.1907	.2363	.2524	.2363	.1907	.1235	.0498
.20	.0521	.1249	.1898	.2333	.2485	.2333	.1898	.1249	.0521
.25	.0536	.1255	.1882	.2298	.2443	.2298	.1882	.1255	.0536
.30	.0547	.1257	.1866	.2266	.2405	.2266	.1866	.1257	.0547
				$I/AL^2 = 1/10000$					
.10	.0425	.1104	.1742	.2180	.2335	.2180	.1742	.1104	.0425
.15	.0483	.1198	.1850	.2293	.2449	.2293	.1850	.1198	.0483
.20	.0512	.1230	.1868	.2297	.2447	.2297	.1868	.1230	.0512
.25	.0531	.1243	.1864	.2277	.2421	.2277	.1864	.1243	.0531
.30	.0544	.1249	.1854	.2253	.2391	.2253	.1854	.1249	.0544
				$I/AL^2 = 1/5000$					
.10	.0371	.0966	.1526	.1912	.2047	.1912	.1526	.0966	.0371
.15	.0454	.1129	.1745	.2164	.2311	.2164	.1745	.1129	.0454
.20	.0496	.1191	.1811	.2227	.2373	.2227	.1811	.1191	.0496
.25	.0520	.1219	.1830	.2235	.2377	.2235	.1830	.1219	.0520
.30	.0536	.1234	.1832	.2226	.2362	.2226	.1832	.1234	.0536

Table XV. Influence Line Coefficients for the Fixed-End Shear at the Left-Hand End of Uniform Semi-Elliptical Arches with Various Values of I/AL^2

$$V^f = \text{shear coefficient} * P$$

r/L \ X	.1	.2	.3	.4	.5	.6	.7	.8	.9
				$I/AL^2 = 0$					
.10	.9658	.8868	.7756	.6431	.5000	.3569	.2244	.1132	.0342
.15	.9619	.8808	.7700	.6398	.5000	.3602	.2300	.1192	.0381
.20	.9586	.8755	.7650	.6369	.5000	.3631	.2350	.1245	.0414
.25	.9558	.8710	.7607	.6343	.5000	.3657	.2393	.1290	.0442
.30	.9535	.8672	.7571	.6322	.5000	.3678	.2429	.1328	.0465
.35	.9517	.8641	.7541	.6304	.5000	.3696	.2459	.1359	.0483
.40	.9502	.8615	.7516	.6288	.5000	.3712	.2484	.1385	.0498
.45	.9490	.8594	.7495	.6276	.5000	.3724	.2505	.1406	.0510
.50	.9480	.8576	.7477	.6265	.5000	.3735	.2523	.1424	.0520
				$I/AL^2 = 1/20000$					
.10	.9657	.8866	.7754	.6430	.5000	.3570	.2246	.1134	.0343
.15	.9618	.8807	.7699	.6398	.5000	.3602	.2301	.1193	.0382
.20	.9585	.8754	.7649	.6368	.5000	.3632	.2351	.1246	.0415
.25	.9557	.8709	.7607	.6343	.5000	.3657	.2393	.1291	.0443
.30	.9535	.8672	.7571	.6321	.5000	.3679	.2429	.1328	.0465
				$I/AL^2 = 1/10000$					
.10	.9656	.8865	.7552	.6429	.5000	.3571	.2248	.1135	.0344
.15	.9617	.8806	.7698	.6397	.5000	.3603	.2302	.1194	.0383
.20	.9584	.8753	.7648	.6367	.5000	.3633	.2352	.1247	.0416
.25	.9556	.8708	.7606	.6342	.5000	.3658	.2394	.1292	.0444
.30	.9534	.8671	.7570	.6321	.5000	.3679	.2430	.1229	.0466
				$I/AL^2 = 1/5000$					
.10	.9655	.8863	.7751	.6428	.5000	.3572	.2249	.1137	.0345
.15	.9616	.8805	.7697	.6397	.5000	.3603	.2303	.1195	.0384
.20	.9583	.8752	.7647	.6466	.5000	.3634	.2353	.1248	.0417
.25	.9555	.8707	.7605	.6342	.5000	.3658	.2395	.1293	.0445
.30	.9533	.8670	.7569	.6320	.5000	.3680	.2431	.1330	.0467

APPENDIX B

DEFINITIONS AND THEOREMS RELATED TO THE CONVERGENCE OF TRANSMISSION MATRICES

Matrix Norms

It is useful to have a single number which gives an overall assessment of the size of a matrix.

The norm of a matrix $[T]$ is any real-valued function $\| T \|$ of the elements of $[T]$ satisfying the four following conditions:

$$\| T \| > 0, \quad \text{unless } [T] = 0 \tag{B.1}$$

$$\| k[T] \| = | k | \| T \| \quad \text{for any scalar } k \tag{B.2}$$

$$\| [T_1] + [T_2] \| \leqslant \| T_1 \| + \| T_2 \| \tag{B.3}$$

$$\| [T_1][T_2] \| \leqslant \| T_1 \| \cdot \| T_2 \| \tag{B.4}$$

From the condition (B.4), it follows that for any positive integer k,

$$\begin{aligned} \| [T]^k \| &\leqslant \| T \| \cdot \| T \| \cdots \| T \| \\ &\leqslant \| T \|^k \end{aligned} \tag{B.5}$$

Consequently, if $\| T \| < 1$, the sequence of norms of power of $[T]$ must vanish in the limit.

Three types of matrix norms are defined as follows:

The first matrix norm

$$\| T \|_1 = \max_i \sum_j t_{ij} \tag{B.6}$$

represents the largest row sum of absolute values.

The second matrix norm

$$\|T\|_2 = \max_j \sum_i t_{ij} \tag{B.7}$$

represents the largest column sum of absolute values.

The third matrix norm,

$$\|T\|_3 = \max\sqrt{\Sigma t_{ij}^2} \tag{B.8}$$

represents the square root of the sum of squares of the elements of a matrix.

These norms can be used to estimate the rate of convergence of matrix sequences and series.

If $\|T\|_3 < 1$, then the limit of $\|T\|^k$ as $k \to \infty$ is zero. It follows that the limit of $[T]^k$ itself is zero, i.e., that each of its elements approaches zero because of the definition of the matrix norm.

Theorem 1. In order that $[T]^k \to 0$, it is necessary and sufficient that all eigenvalues of the matrix $[T]$ be less than one in modulus.

It is clear that for any fixed non-singular matrix $[R]$ the matrices $[T]^k$ and $([R]^{-1}[T][R])^k = [R]^{-1}[T]^k[R]$ do, or fail to, approach zero simultaneously. Therefore, it suffices to prove the validity of the theorem for a Jordan canonical matrix. Moreover, in order that a sequence of quasi-diagonal matrices of the same structure approach zero, it is necessary and sufficient that the sequence of separate boxes converges to zero. Thus, it is desired to establish the condition of convergence to zero only for matrices $[J]^k$, where $[J]$ is a Jordan canonical box. Let

$$[J] = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{bmatrix} \tag{B.9}$$

It is readily found that for $m \geqslant k-1$

$$[J]^k = \begin{bmatrix} \lambda^k & 0 & 0 \\ \binom{k}{1}\lambda^{k-1} & \lambda^k & 0 \\ \cdot\cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot \\ \binom{k}{m-1}\lambda^{k-m+1} & \binom{k}{m-2}\lambda^{k-m+2} \; \cdots & \lambda^k \end{bmatrix} \tag{B.10}$$

where m is the order of the box, and

$$\binom{k}{j} = \frac{k(k-1)\cdots(k-j+1)}{j!}$$

For $[J]^k$ to converge to zero it is necessary that $|\lambda| < 1$, since λ^k are the diagonal elements of $[J]^k$.

But this condition is also sufficient, since under this condition

$$\frac{k(k-1)\cdots(k-j+1)}{j!}\lambda^{k-j} \rightarrow 0$$

for $k \rightarrow \infty$, for all $j = 1, \ldots, m-2$.

The given conditions in theorem 1 are inconvenient for verifying this, since they require a knowledge of the eigenvalues of matrix [T]. Therefore, it is necessary to establish certain simpler sufficiency conditions in order that $[T]^k \rightarrow 0$.

<u>Theorem 2.</u> In order that $[T]^k \rightarrow 0$, it is sufficient that at

least one of norms of [T] be less than one.

In view of the requirement for the norm

$$\|T^k\| \leqslant \|T^{k-1}\|\cdot\|T\| \leqslant \|T\|^k$$

Therefore if $\|T\| < 1$, then $\|T^k\| \to 0$ and, in view of what was said above, $[T]^k \to 0$.

Comparing theorems 1 and 2, the following conclusion is drawn:

<u>Theorem 3</u>. The modulus of every eigenvalue of a matrix does not exceed any of its norms.

Let $\|T\| = t$. Consider the matrix $[S] = \frac{1}{t + \varepsilon}[T]$, where ε is any positive number. Then

$$\|S\| = \frac{t}{t + \varepsilon} < 1$$

Consequently, $[S]^k \to 0$ for $k \to \infty$. In view of theorem 1, its eigenvalues will be less than one in modulus; but it is obvious that the eigenvalues of matrix [S] are equal to $\frac{1}{\lambda_i(t + \varepsilon)}$, where λ_i are eigenvalues of matrix [T]. Thus $|\lambda_i|/(t - \varepsilon) < 1$; that is, $|\lambda_i| < t + \varepsilon$; ε is taken as small as desired.

<u>Theorem 4</u>. For the series

$$[I] + [T] + [T]^2 + [T]^3 + \cdots + [T]^k \qquad (B.11)$$

to converge, it is necessary and sufficient that $[T]^k \to 0$ for $k \to \infty$. In this case the sum of the series (B.11) is equal to $\left[[I] - [T]\right]^{-1}$.

The necessary of this condition is obvious. It will show that it is also sufficient. Let $[T]^k \to 0$. On the basis of theorem 1 all eigenvalues of matrix [T] are less than one in modulus. Consequently

$\left|[I] - [T]\right| \neq 0$ and therefore $\left[[I] - [T]\right]^{-1}$ exists.

Consider the relation

$$\left([I] + [T] + [T]^2 + \ldots + [T]^k\right)\left([I] - [T]\right) = [I] - [T]^{k+1}$$

Post-multiplication by $\left[[I] - [T]\right]^{-1}$ yields

$$[I] + [T] + \ldots + [T]^k = \left[[I] - [T]\right]^{-1} - [T]^{k+1}\left[[I] - [T]\right]^{-1}$$

The second term on the right tends to zero when $k \longrightarrow \infty$, and therefore

$$[I] + [T] + [T]^2 + \ldots + [T]^k \longrightarrow \left[[I] - [T]\right]^{-1}$$

On the basis of theorem 1 the necessary and sufficient condition for the convergence of the infinite matrix series (B.11) is the inequality $|\lambda_i| < 1$ for all eigenvalues of matrix $[T]$. A sufficient condition of convergence, on the basis of theorem 2, is the inequality $\|T\| < 1$ for at least one of the norms. For satisfying this condition the following estimate for the rate of convergence of series (B.11) may be used.

Theorem 5. If $\|T\| < 1$, then

$$\left\|\left[[I] - [T]\right]^{-1} - [I] + T + \ldots + [T]^k\right\| \leqslant \frac{\|T\|^{k+1}}{1 - \|T\|} \qquad \text{(B.12)}$$

It is evident that

$$\left[[I] - [T]\right]^{-1} - [I] + [T] + \ldots + [T]^k = [T]^{k+1} + [T]^{k+2} + \ldots$$

This implies that

$$\left\|\left[[I] - [T]\right]^{-1} - [I] + [T] + \ldots + [T]^k\right\| \leqslant \|T\|^{k+1} + \|T\|^{k+2} + \ldots$$

$$= \frac{\|T\|^{k+1}}{1 - \|T\|}$$

The theorem is proved.

APPENDIX C

MECHANICAL ANALYSIS OF STRUCTURAL MODELS BY USING BEGGS DEFORMETER

Principle of Mechanical Analysis

Mechanical analysis is based on the Muller-Breslau principle. This principle is ideal for structural model studies wherein the deformations become measured quantities instead of computed quantities.

Fig. 28 illustrates the principle underlying the determination of the reactions for a continuous arch frame fixed at the footings by measuring relative displacements on the model. It is desired to find the reactions at point 2′, for example, due to a load P acting as shown.

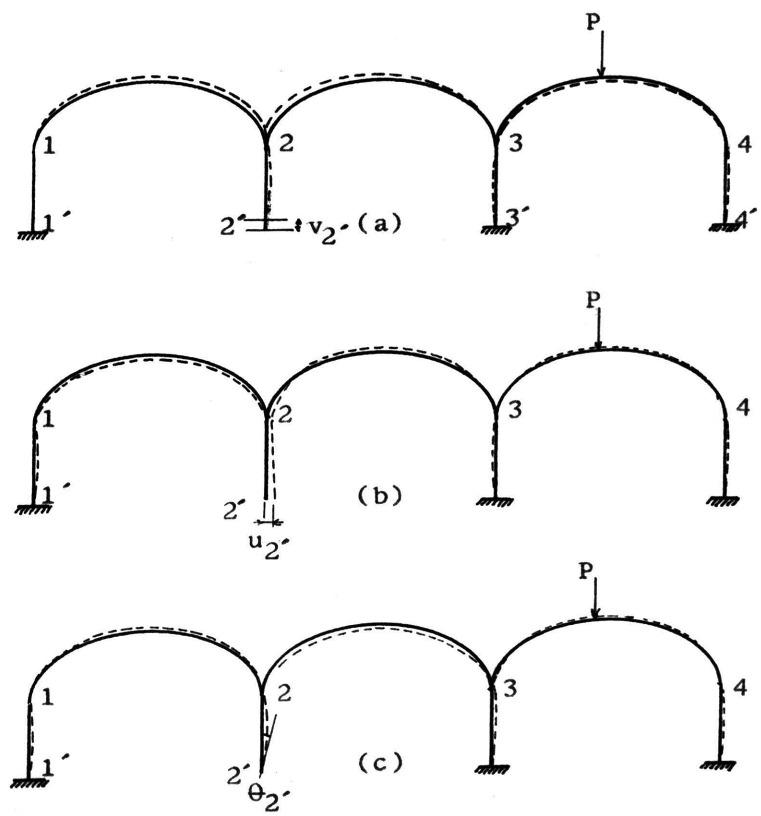


Fig. 28. Determination of Reactions by Mechanical Analysis

A step by step procedure of mechanical analysis for determining components of the reaction at point 2′ is as follows:

Move point 2′ vertically a known amount $v_{2'}$ without permitting rotation (Fig. 28.a). Measure carefully the corresponding deflection v_P at the loaded point. It follows that the vertical component of the reaction is

$$V_{2'} = P \frac{v_P}{v_{2'}}$$

Restore the model to its unstrained position. Move point 2′ horizontally a known amount $u_{2'}$ without permitting rotation (Fig. 28.c). Measure the corresponding displacement v_P at the loaded point; then the horizontal component of the reaction is computed as follows:

$$H_{2'} = P \frac{v_P}{u_{2'}}$$

Restore the model in its unstrained position again. Rotate point 2′ without permitting linear displacements (Fig. 28.c). Measure the rotation $\theta_{2'}$ and the corresponding deflection v_P at the loaded point. The moment $M_{2'}$ at point 2′ is

$$M_{2'} = \ell P \frac{v_P}{\theta_{2'}}$$

in which ℓ is the scale ratio of the model.

Thus, components of the reaction at point 2′ may be expressed in terms of the load P and relative displacements.

<u>Beggs Deformeter</u>

The Beggs deformeter was introduced in 1922 by the late Professor G.E. Beggs of Princeton University. The deformeter consists of several gages and one or more microscopes.

The principle of the Beggs deformeter gages is illustrated in Fig. 29. The gages are placed at each reaction of the model and may be used either to impress a known displacement or to supply the reaction components.

(a) Thrust plugs

(b) Shear plugs

(c) Moment plugs

Fig. 29. Beggs Deformeter Gages

The gage clamp as shown in Fig. 30, is constructed for the purpose of attaching the model to the table and for producing displacements of known value in the model at the reaction points. The fixed bar f is fastened to the board on which the model is mounted and the reaction point of the model is fastened to the movable bar m which is held against the gage plugs by means of a spring connection between f and m.

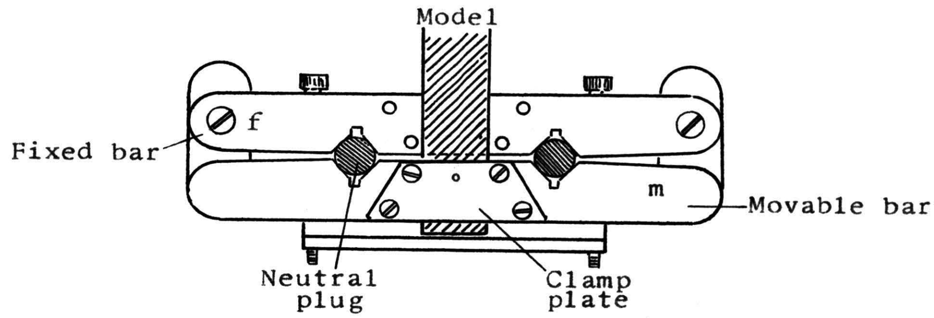


Fig. 30. Detail of a Beggs Gage Clamp

In the unstrained position of the model, neutral plugs are in place. These may be removed by spreading the bars f and m by means of small wedges.

To produce a known vertical displacement of the reaction point, the neutral plugs are removed and smaller diameter plugs of like size are inserted. A reading on the model at the point of application of the load is then taken. The small plugs are then removed from the deformeter gages and the larger diameter plugs are inserted, the difference in diameter of the small and large plugs being known vertical displacement. A second reading is then taken at the point of application of the load and the difference of the two readings gives the displacement at the loaded point.

Horizontal displacements at the reaction points are produced by means of "horizontal thrust plugs" as indicated in Fig. 29.a. Rotation of the reaction point is produced by interchanging a small size plug and a large size plug in the sockets of the deformeter gages as indicated in Fig. 29.c.

The arbitrarily imposed displacements at reaction points of the model are accomplished by means of deformeter gages capable of producing very small deflections with an accuracy to 1/40000 of an inch; the corresponding small displacements at the points of application of the loads are measured by means of filar micrometer microscopes.

The reading instruments are microscopes set in heavy frames for stability. A scale is engraved on glass within the instrument and the field of view. Cross-hairs on another glass are caused to move along this scale by a micrometer movement. The system is so arranged that one division on the micrometer index represents a movement of an observed point on the model of about one six-thousandth of an inch.

Assume that a model for a three-span continuous frame with rigid column bases is to be mounted for analysis. A smooth table top or drawing board is placed in a horizontal position, and on this board the positions of the column bases are carefully located. The four gage clamps required, one for each column base, are carefully screwed to the board in the proper location with neutral plugs in each gage. Orientation marks are provided on the gages for accurate mounting. The model columns are then clamped to the movable bar of each gage with the clamp plate and screws; these columns are usually made 1 or 2 in. longer than scale length to permit mounting in the gages. The model must be free to move at all points between gages, and this is made possible by supporting it on small ball bearings which roll on small pieces of plate glass. Lead weights are usually placed on top to prevent buckling.

If an analysis is desired for column bases pinned, holes are drilled in the base of the model columns at points to correspond to the actual pin location in the prototype. The clamp plates are removed from the movable bars and pins are inserted in the pin holes. The holes in the model columns, which must be drilled for a smooth fit, permit the mounting of the model on these pins.

It is to be noted that no actual loads are applied to the model; the process being only the measurement of related displacements, the ratio of which is the ratio of load to reaction component.

APPENDIX D

COMPUTER PROGRAMS FOR ANALYSIS OF CONTINUOUS CURVILINEAR STRUCTURES BY INFINITE MATRIX SERIES METHODS

Identifiers Used in Computer Programs

I,J	Index numbers that identify the joint
K	Index number that identifies the loading condition
NI	Number of interior joints of a continuous system
NLC	Number of loading conditions
E(I,J)	Identity matrix
U(I,K)	Joint displacement matrix
S(I,J)	Matrix sum of the infinite matrix series
UMO(I,K)	Initial unbalanced moment matrix due to external loads
UMDO(I,J)	Initial unbalanced moment matrix due to unit displacements at joints
UHO(I,K)	Initial unbalanced thrust matrix due to external loads
UHDO(I,J)	Initial unbalanced thrust matrix due to unit displacements at joints
UFO(I,K)	Initial unbalanced generalized force matrix
UMS(I,K)	Matrix sum of all moments relaxed at joints due to external loads
UMDS(I,J)	Matrix sum of all moments relaxed at joints due to unit displacements
UFS(I,K)	Matrix sum of all generalized forces relaxed at joints
UMT(I,K)	Total unbalanced moment matrix due to external loads
UMDT(I,J)	Total unbalanced moment matrix due to unit displacements at joints
UFT(I,K)	Total unbalanced generalized force matrix

In the following identifiers, a suffix L, R or C has been used in computer programs to indicate that the quantity occurs at the left, right or column side of all interior joints of a continuous system.

S11R(I), S11L(I), S11C(I)	Thrust stiffness factors
S13R(I), S13L(I), S13C(I)	Thrust-moment stiffness factors
S31R(I), S31L(I), S31C(I)	Moment-thrust stiffness factors
S23R(I), S23L(I), S23C(I)	Shear-moment stiffness factors
S33R(I), S33L(I), S33C(I)	Moment stiffness factors
COFR(I), COFL(I), COFC(I)	Moment carry-over factors
TIFR(I), TIFL(I), TIFC(I)	Thrust-induction factors
SPNR(I), SPNL(I), SPNC(I)	Spans
D11R(I), D11L(I), D11C(I)	Thrust distribution factors
D13R(I), D13L(I), D13C(I)	Thrust-moment distribution factors
D31R(I), D31L(I), D31C(I)	Moment-thrust distribution factors
D33R(I), D33L(I), D33C(I)	Generalized moment distribution factors
T11R(I), T11L(I)	Thrust transmission factors
T13R(I), T13L(I)	Thrust-moment transmission factors
T31R(I), T31L(I)	Moment-thrust transmission factors
T33R(I), T33L(I)	Generalized moment transmission factors
DR(I,J), DL(I,J), DC(I,J)	Distribution matrices
TT(I,J), TR(I,J), TL(I,J)	Transposed transmission matrices
DTR(I,J), DTL(I,J)	Distribution-transmission matrices
FEMR(I,K), FEML(I,K), FEMC(I,K)	Fixed-end moment matrices due to external loads
FEHR(I,K), FEHL(I,K), FEHC(I,K)	Fixed-end thrust matrices due to external loads
FFR(I,K), FFL(I,K), FFC(I,K)	Fixed-end generalized force matrices due to external loads
RMR(I,K), RML(I,K), RMC(I,K)	Rotational moment matrices due to external loads
RHR(I,K), RHL(I,K), RHC(I,K)	Rotational thrust matrices due to external loads
RMDR(I,J), RMDL(I,J), RMDC(I,J)	Rotational moment matrices due to unit displacements at joints
RHDR(I,J), RHDL(I,J), RHDC(I,J)	Rotational thrust matrices due to unit displacements at joints

EMR(I,K), EML(I,K), EMC(I,K)	End moment matrices due to external loads
EHR(I,K), EHL(I,K), EHC(I,K)	End thrust matrices due to external loads
EMDR(I,J), EMDL(I,J), EMDC(I,J)	End moment matrices due to unit displacements at joints
EHDR(I,J), EHDL(I,J), EHDC(I,J)	End thrust matrices due to unit displacements at joints
FEMDR(I,J), FEMDL(I,J), FEMDC(I,J)	Fixed-end moment matrices due to unit displacements at joints
FEHDR(I,J), FEHDL(I,J), FEHDC(I,J)	Fixed-end thrust matrices due to unit displacements at joints
EMADR(I,J), EMADL(I,J), EMADC(I,J)	End moment matrices due to actual displacements at joints
EHADR(I,J), EHADL(I,J), EHADC(I,J)	End thrust matrices due to actual displacements at joints
DCFR(I,K), DCFL(I,K), DCFC(I,K)	Distributed and carry-over generalized force matrices
FMR(I,K), FML(I,K), FMC(I,K)	Final moment matrices
FHR(I,K), FHL(I,K), FHC(I,K)	Final thrust matrices
FR(I,K), FL(I,K), FC(I,K)	Final generalized force matrices

Listing of Computer Programs

```
C ********************************************************
C *                                                      *
C *    ANALYSIS OF CONTINUOUS CURVILINEAR STRUCTURES     *
C *                                                      *
C *    BY RESTRAINED INFINITE MATRIX SERIES METHOD       *
C *                                                      *
C ********************************************************
C
C
      DIMENSION S11R(5),S11L(5),S11C(5),S33R(5),S33L(5),
     &  S33C(5),S13R(5),S13L(5),S13C(5),S23R(5),S23L(5),
     &  S23C(5),COFR(5),COFL(5),COFC(5),TIFR(5),TIFL(5),
     &  TIFC(5),SPNR(5),SPNL(5),SPNC(5),SS33(5),D33R(5),
     &  D33L(5),D33C(5),T33R(5),T33L(5)
      DIMENSION DR(5,5),DL(5,5),DC(5,5),TR(5,5),TL(5,5),
     &   TT(5,5),DTR(5,5),DTL(5,5),E(5,5),Q(5,5),S(5,5),
     &    TS(5,5),UMO(5,3),UMDO(5,5),UMS(5,3),UMDS(5,5),
     & UMT(5,3),UMDT(5,5),UHO(5,3),UHDO(5,5),U(5,3)
      DIMENSION FEMR(5,3),FEML(5,3),FEMC(5,3),FEHR(5,3),
     &FEHL(5,3),FEHC(5,3),FEMDR(5,5),FEMDL(5,5),FEMDC(5,
     &5),FEHDR(5,5),FEHDL(5,5),FEHDC(5,5)
      DIMENSION  RMR(5,3),RML(5,3),RMC(5,3),RHR(5,3),RHL
     & (5,3),RHC(5,3),RMDR(5,5),RMDL(5,5),RMDC(5,5),RHDR
     &  (5,5),RHDL(5,5),RHDC(5,5),EMADR(5,5),EMADL(5,5),
     &  EMADC(5,5),FMR(5,3),FML(5,3),FMC(5,3)
      DIMENSION  EMR(5,3),EML(5,3),EMC(5,3),EHR(5,3),EHL
     & (5,3),EHC(5,3),EMDR(5,5),EMDL(5,5),EMDC(5,5),EHDR
     &  (5,5),EHDL(5,5),EHDC(5,5),EHADR(5,5),EHADL(5,5),
     &  EHADC(5,5),FHR(5,3),FHL(5,3),FHC(5,3)
C
C
  100 READ (1,200) NI,NLC
C
      IF (NI .EQ. 0) GO TO 300
      WRITE (3,210)
      NIM1=NI-1
C
C     READ AND WRITE OUT PROPERTIES OF SEGMENTAL ARCHES
C
      READ (1,201) (S11R(I),S33R(I),S13R(I),S23R(I),
     &SPNR(I),S11L(I),S33L(I),S13L(I),S23L(I),SPNL(I),
     &S11C(I),S33C(I),S13C(I),S23C(I),SPNC(I),I=1,NI)
C
      WRITE (3,211)
      WRITE (3,202)  (I,S11R(I),S11L(I),S11C(I),I=1,NI)
      WRITE (3,212)
      WRITE (3,202)  (I,S33R(I),S33L(I),S33C(I),I=1,NI)
      WRITE (3,213)
      WRITE (3,202)  (I,S13R(I),S13L(I),S13C(I),I=1,NI)
      WRITE (3,214)
```

```
      WRITE (3,202) (I,S23R(I),S23L(I),S23C(I),I=1,NI)
      WRITE (3,215)
      WRITE (3,202) (I,SPNR(I),SPNL(I),SPNC(I),I=1,NI)
C
      DO 108 I=1,NI
      IF (S33L(I)) 102,101,102
  101 COFL(I)=0.
      TIFL(I)=0.
      GO TO 103
  102 COFL(I)=(-S33L(I)-S23L(I)*SPNL(I))/S33L(I)
      TIFL(I)=(S13L(I)/S33L(I))*(1.+COFL(I))/(1.-COFL(I)**2)
  103 IF (S33R(I)) 105,104,105
  104 COFR(I)=0.
      TIFR(I)=0.
      GO TO 106
  105 COFR(I)=(-S33R(I)-S23R(I)*SPNR(I))/S33R(I)
      TIFR(I)=(S13R(I)/S33R(I))*(1.+COFR(I))/(1.-COFR(I)**2)
  106 COFC(I)=0.5*S23C(I)
C
C     S23C(I) = 0 : THE OTHER END OF MEMBER IS HINGED
C     S23C(I) = 1 : THE OTHER END OF MEMBER IS FIXED
C
C     DISTRIBUTION FACTORS
C
      SS33(I)=S33R(I)+S33L(I)+S33C(I)
      D33R(I)=-S33R(I)/SS33(I)
      D33L(I)=-S33L(I)/SS33(I)
      D33C(I)=-S33C(I)/SS33(I)
C
C     TRANSMISSION FACTORS
C
      T33R(I)=COFR(I)*D33R(I)
  108 T33L(I)=COFL(I)*D33L(I)
C
C     DISTRIBUTION MATRICES
C
      DO 110 I=1,NI
      DO 110 J=1,NI
      DR(I,J)=0.
      DL(I,J)=0.
      DC(I,J)=0.
      DR(I,I)=D33R(I)
      DL(I,I)=D33L(I)
  110 DC(I,I)=D33C(I)
C
C     TRANSMISSION MATRICES
C
      DO 115 I=1,NI
      DO 115 J=1,NI
      TR(I,J)=0.
  115 TL(I,J)=0.
      DO 116 I=2,NI
```

```
  116 TR(I,I-1)=T33R(I-1)
      DO 117 I=1,NIM1
  117 TL(I,I+1)=T33L(I+1)
C
      DO 118 I=1,NI
      DO 118 J=1,NI
      TT(I,J)=TR(I,J)+TL(I,J)
      DTR(I,J)=DR(I,J)+TL(I,J)
  118 DTL(I,J)=DL(I,J)+TR(I,J)
C
      WRITE (3,216)
      DO 119 I=1,NI
  119 WRITE (3,205) (TR(I,J),J=1,NI)
      WRITE (3,217)
      DO 120 I=1,NI
  120 WRITE (3,205) (TL(I,J),J=1,NI)
      WRITE (3,218)
      DO 121 I=1,NI
  121 WRITE (3,205) (TT(I,J),J=1,NI)
      WRITE (3,219)
      DO 122 I=1,NI
  122 WRITE (3,205) (DTR(I,J),J=1,NI)
      WRITE (3,220)
      DO 123 I=1,NI
  123 WRITE (3,205) (DTL(I,J),J=1,NI)
      WRITE (3,221)
      DO 124 I=1,NI
  124 WRITE (3,205) (DC(I,J),J=1,NI)
C
C     FIXED-END MOMENT AND FIXED-END THRUST MATRICES
C     DUE TO UNIT DISPLACEMENT AT JOINTS
C
      DO 130 I=1,NI
      DO 130 J=1,NI
      FEMDR(I,J)=0.
      FEMDL(I,J)=0.
      FEMDC(I,J)=0.
      FEHDR(I,J)=0.
      FEHDL(I,J)=0.
      FEHDC(I,J)=0.
      FEMDL(I,I)=S13L(I)
      FEMDR(I,I)=S13R(I)
      FEMDC(I,I)=S13C(I)
      FEHDL(I,I)=S11L(I)
      FEHDR(I,I)=S11R(I)
  130 FEHDC(I,I)=S11C(I)
      DO 131 I=1,NIM1
      FEMDR(I,I+1)=-S13R(I)
  131 FEHDR(I,I+1)=-S11R(I)
      DO 132 I=2,NI
      FEMDL(I,I-1)=-S13L(I)
  132 FEHDL(I,I-1)=-S11L(I)
```

```
C     READ AND WRITE OUT EXTERNAL LOADS
C
      DO 140 K=1,NLC
  140 READ (1,204) (FEHR(I,K),FEHL(I,K),FEHC(I,K),
     &FEMR(I,K),FEML(I,K),FEMC(I,K),I=1,NI)
      DO 141 K=1,NLC
      WRITE (3,222) K
  141 WRITE (3,206) (I,FEHR(I,K),FEHL(I,K),FEHC(I,K),
     &FEMR(I,K),FEML(I,K),FEMC(I,K),I=1,NI)
C
C     INITIAL UNBALANCED MOMENTS AND THRUSTS
C
      DO 145 I=1,NI
      DO 144 K=1,NLC
  144 UMO(I,K)=FEML(I,K)+FEMR(I,K)+FEMC(I,K)
      DO 145 J=1,NI
  145 UMDO(I,J)=FEMDL(I,J)+FEMDR(I,J)+FEMDC(I,J)
C
C     MATRIX SUM OF THE INFINITE MATRIX SERIES IN T
C     (INVERSE OF MATRIX Q)
C
      DO 150 I=1,NI
      DO 150 J=1,NI
      E(I,J)=0.
      E(I,I)=1.
  150 Q(I,J)=E(I,J)-TT(I,J)
      DO 151 I=1,NI
      DO 151 J=1,NI
  151 S(I,J)=E(I,J)
      DET=1.
      DO 157 K=1,NI
      DET=-DET
      DET=Q(K,K)*DET
      QDIV=Q(K,K)
      DO 152 J=1,NI
      Q(K,J)=Q(K,J)/QDIV
  152 S(K,J)=S(K,J)/QDIV
      DO 156 I=1,NI
      QMULT=Q(I,K)
      IF (I-K) 154,156,154
  154 DO 155 J=1,NI
      Q(I,J)=Q(I,J)-QMULT*Q(K,J)
  155 S(I,J)=S(I,J)-QMULT*S(K,J)
  156 CONTINUE
  157 CONTINUE
      WRITE (3,223)
      DO 158 I=1,NI
  158 WRITE (3,205) (S(I,J),J=1,NI)
C
C     PRODUCT OF  T & S
C
C
```

```
      DO 160 I=1,NI
      DO 160 J=1,NI
      TS(I,J)=0.
      DO 160 L=1,NI
  160 TS(I,J)=TS(I,J)+TT(I,L)*S(L,J)
      WRITE (3,224)
      DO 161 I=1,NI
  161 WRITE (3,205) (TS(I,J),J=1,NI)
C
C     SUMS OF ALL MOMENTS RELAXED AT JOINTS
C     (INVERSE OF MATRIX Q)
C
      DO 162 I=1,NI
      DO 162 K=1,NLC
      UMS(I,K)=0.
      DO 162 L=1,NI
  162 UMS(I,K)=UMS(I,K)+TS(I,L)*UMO(L,K)
      DO 163 I=1,NI
      DO 163 J=1,NI
      UMDS(I,J)=0.
      DO 163 L=1,NI
  163 UMDS(I,J)=UMDS(I,J)+TS(I,L)*UMDO(L,J)
      WRITE (3,225)
C
      DO 146 I=1,NI
  146 WRITE (3,205) (UMS(I,K),K=1,NLC)
      WRITE (3,226)
      DO 147 I=1,NI
  147 WRITE (3,205) (UMDS(I,J),J=1,NI)
C
C     TOTAL UNBALANCED MOMENTS AT JOINTS
C
      DO 165 I=1,NI
      DO 164 K=1,NLC
  164 UMT(I,K)=UMO(I,K)+UMS(I,K)
      DO 165 J=1,NI
  165 UMDT(I,J)=UMDO(I,J)+UMDS(I,J)
C
C     ROTATIONAL MOMENTS
C
      DO 166 I=1,NI
      DO 166 K=1,NLC
      RMR(I,K)=0.
      RML(I,K)=0.
      RMC(I,K)=0.
      DO 166 L=1,NI
      RMR(I,K)=RMR(I,K)+DTR(I,L)*UMT(L,K)
      RML(I,K)=RML(I,K)+DTL(I,L)*UMT(L,K)
  166 RMC(I,K)=RMC(I,K)+DC(I,L)*UMT(L,K)
      DO 167 I=1,NI
      DO 167 J=1,NI
      RMDR(I,J)=0.
```

```
      RMDL(I,J)=0.
      RMDC(I,J)=0.
      DO 167 L=1,NI
      RMDR(I,J)=RMDR(I,J)+DTR(I,L)*UMDT(L,J)
      RMDL(I,J)=RMDL(I,J)+DTL(I,L)*UMDT(L,J)
  167 RMDC(I,J)=RMDC(I,J)+DC(I,L)*UMDT(L,J)
C
C     THRUSTS DUE TO ROTATION
C
      DO 173 K=1,NLC
      RHL(1,K)=TIFL(1)*RML(1,K)*(1.-COFL(1))
      RHR(NI,K)=TIFR(NI)*RMR(NI,K)*(1.-COFR(NI))
      DO 170 I=1,NIM1
  170 RHR(I,K)=TIFR(I)*(RMR(I,K)-RML(I+1,K))
      DO 171 I=2,NI
  171 RHL(I,K)=TIFL(I)*(RML(I,K)-RMR(I-1,K))
      DO 172 I=1,NI
  172 RHC(I,K)=-RMC(I,K)*(1.+COFC(I))/SPNC(I)
  173 CONTINUE
      DO 178 J=1,NI
      RHDL(1,J)=TIFL(1)*RMDL(1,J)*(1.-COFL(1))
      RHDR(NI,J)=TIFR(NI)*RMDR(NI,J)*(1.-COFR(NI))
      DO 175 I=1,NIM1
  175 RHDR(I,J)=TIFR(I)*(RMDR(I,J)-RMDL(I+1,J))
      DO 176 I=2,NI
  176 RHDL(I,J)=TIFL(I)*(RMDL(I,J)-RMDR(I-1,J))
      DO 177 I=1,NI
  177 RHDC(I,J)=-RMDC(I,J)*(1.+COFC(I))/SPNC(I)
  178 CONTINUE
C
C     END MOMENTS AND END THRUSTS
C
      DO 181 I=1,NI
      DO 180 K=1,NLC
      EMR(I,K)=FEMR(I,K)+RMR(I,K)
      EML(I,K)=FEML(I,K)+RML(I,K)
      EMC(I,K)=FEMC(I,K)+RMC(I,K)
      EHR(I,K)=FEHR(I,K)+RHR(I,K)
      EHL(I,K)=FEHL(I,K)+RHL(I,K)
  180 EHC(I,K)=FEHC(I,K)+RHC(I,K)
      DO 181 J=1,NI
      EMDR(I,J)=FEMDR(I,J)+RMDR(I,J)
      EMDL(I,J)=FEMDL(I,J)+RMDL(I,J)
      EMDC(I,J)=FEMDC(I,J)+RMDC(I,J)
      EHDR(I,J)=FEHDR(I,J)+RHDR(I,J)
      EHDL(I,J)=FEHDL(I,J)+RHDL(I,J)
  181 EHDC(I,J)=FEHDC(I,J)+RHDC(I,J)
C
C     UNBALALANCED THRUSTS AT JOINTS
C
      DO 183 I=1,NI
      DO 182 K=1,NLC
```

```
  182 UHO(I,K)=EHL(I,K)+EHR(I,K)+EHC(I,K)
      DO 183 J=1,NI
  183 UHDO(I,J)=EHDR(I,J)+EHDL(I,J)+EHDC(I,J)
      WRITE (3,227)
      DO 184 I=1,NI
  184 WRITE (3,205) ((UHDO(I,J),J=1,NI),UHO(I,K),K=1,NLC)
C
C     SOLUTION OF SYSTEM OF SIMULTANEOUS EQUATIONS
C     FOR UNKNOWN HORIZONTAL DISPLACEMENTS
C
      DEL=1.0
      DO 191 I=1,NI
      DEL=-DEL
      DEL=UHDO(I,I)*DEL
      DIV=UHDO(I,I)
      DO 185 K=1,NLC
  185 UHO(I,K)=UHO(I,K)/DIV
      DO 186 J=1,NI
  186 UHDO(I,J)=UHDO(I,J)/DIV
      DO 190 M=1,NI
      UHDM=UHDO(M,I)
      IF (M-I) 187,190,187
  187 DO 188 K=1,NLC
  188 UHO(M,K)=UHO(M,K)-UHDM*UHO(I,K)
      DO 189 J=1,NI
  189 UHDO(M,J)=UHDO(M,J)-UHDM*UHDO(I,J)
  190 CONTINUE
  191 CONTINUE
C
C     ACTUAL JOINT DISPLACEMENTS
C
      DO 192 I=1,NI
      DO 192 K=1,NLC
  192 U(I,K)=-UHO(I,K)
      WRITE (3,228)
      DO 193 I=1,NI
  193 WRITE (3,205) (U(I,K),K=1,NLC)
C
C     ACTUAL END MOMENTS AND THRUSTS
C
      DO 194 I=1,NI
      DO 194 K=1,NLC
      EMADR(I,K)=0.
      EMADL(I,K)=0.
      EMADC(I,K)=0.
      EHADR(I,K)=0.
      EHADL(I,K)=0.
      EHADC(I,K)=0.
      DO 194 L=1,NI
      EMADR(I,K)=EMADR(I,K)+EMDR(I,L)*U(L,K)
      EMADL(I,K)=EMADL(I,K)+EMDL(I,L)*U(L,K)
      EMADC(I,K)=EMADC(I,K)+EMDC(I,L)*U(L,K)
```

```
      EHADR(I,K)=EHADR(I,K)+EHDR(I,L)*U(L,K)
      EHADL(I,K)=EHADL(I,K)+EHDL(I,L)*U(L,K)
  194 EHADC(I,K)=EHADC(I,K)+EHDC(I,L)*U(L,K)
C
C     FINAL MOMENTS AND FINAL THRUSTS
C
      DO 195 I=1,NI
      DO 195 K=1,NLC
      FMR(I,K)=EMR(I,K)+EMADR(I,K)
      FML(I,K)=EML(I,K)+EMADL(I,K)
      FMC(I,K)=EMC(I,K)+EMADC(I,K)
      FHR(I,K)=EHR(I,K)+EHADR(I,K)
      FHL(I,K)=EHL(I,K)+EHADL(I,K)
  195 FHC(I,K)=EHC(I,K)+EHADC(I,K)
      WRITE (3,229)
      DO 196 I=1,NI
      WRITE (3,205) (FHR(I,K),K=1,NLC)
  196 WRITE (3,205) (FMR(I,K),K=1,NLC)
      WRITE (3,230)
      DO 197 I=1,NI
      WRITE (3,205) (FHL(I,K),K=1,NLC)
  197 WRITE (3,205) (FML(I,K),K=1,NLC)
      WRITE (3,231)
      DO 198 I=1,NI
      WRITE (3,205) (FHC(I,K),K=1,NLC)
  198 WRITE (3,205) (FMC(I,K),K=1,NLC)
C
  200 FORMAT (2I5)
  201 FORMAT (5F15.9)
  202 FORMAT (I10,3F20.6)
  204 FORMAT (6F10.4)
  205 FORMAT (8F12.6/(8F12.6))
  206 FORMAT (I5,6F12.3)
  210 FORMAT (1H1,'                                ',/,10X,
     &'ANALYSIS OF CONTINUOUS CURVILINEAR STRUCTURES',//,
     &11X,'BY RESTRAINED INFINITE MATRIX SERIES METHOD'//
     &///)
  211 FORMAT (//,7X,'JOINT',13X,'R',19X,'L',19X,'C',
     &//,5X,'K11'/)
  212 FORMAT (//,'      K33'/)
  213 FORMAT (//,'      K13'/)
  214 FORMAT (//,'      K23'/)
  215 FORMAT (//,'      SPAN'/)
  216 FORMAT (///,10X,'RIGHT-SIDE TRANSMISSION MATRIX'//)
  217 FORMAT (///,10X,'LEFT-SIDE TRANSMISSION MATRIX'//)
  218 FORMAT (///,10X,
     &'THE TRANSPOSE OF THE TRANSMISSION MATRIX'//)
  219 FORMAT (///,10X,
     &'RIGHT-SIDE DISTRIBUTION-TRANSMISSION MATRIX'//)
  220 FORMAT (///,10X,
     &'LEFT-SIDE DISTRIBUTION-TRANSMISSION MATRIX'//)
  221 FORMAT (///,10X,'COLUMN-SIDE DISTRIBUTION MATRIX'//)
```

```
  222 FORMAT (///,10X,'LOADING CONDITION NUMBER ',I4,
     &//,2X,'JOINT',5X,'FEHR',9X,'FEHL',9X,'FEHC',9X,
     &'FEMR',9X,'FEML',9X,'FEMC'/)
  223 FORMAT (///,10X,
     &'MATRIX SUM OF THE INFINITE MATRIX SERIES IN T'//)
  224 FORMAT (///,10X,'PRODUCT OF  T & S '//)
  225 FORMAT (///,10X,'SUMS OF UNBALANCED MOMENTS',//,
     &10X,'DUE TO EXTERNAL LOADS'//)
  226 FORMAT (///,10X,
     &'DUE TO UNIT DISPLACEMENT AT JOINTS'//)
  227 FORMAT (///,10X,'EQUILIBRIUM EQUATIONS AT JOINTS'//)
  228 FORMAT (///,10X,'ACTUAL DISPLACEMENTS'//)
  229 FORMAT (1H1,10X,
     &'RIGHT SIDE FINAL THRUSTS AND MOMENTS'//)
  230 FORMAT (///,10X,
     &'LEFT SIDE FINAL THRUSTS AND MOMENTS'//)
  231 FORMAT (///,10X,
     &'COLUMN SIDE FINAL THRUSTS AND MOMENTS'//)
      GO TO 100
  300 STOP
      END

/DATA
```

```
C ********************************************************
C *                                                      *
C *    ANALYSIS OF CONTINUOUS CURVILINEAR STRUCTURE      *
C *                                                      *
C *    BY GENERALIZED INFINITE MATRIX SERIES METHOD      *
C *                                                      *
C ********************************************************
C
C
      DIMENSION S11R(11),S11L(11),S11C(11),SS11(11),
     & D11R(11),D11L(11),D11C(11),T11R(11),T11L(11)
      DIMENSION S33R(11),S33L(11),S33C(11),SS33(11),
     & D33R(11),D33L(11),D33C(11),T33R(11),T33L(11)
      DIMENSION S13R(11),S13L(11),S13C(11),SS13(11),
     & D13R(11),D13L(11),D13C(11),T13R(11),T13L(11)
      DIMENSION S31R(11),S31L(11),S31C(11),SS31(11),
     & D31R(11),D31L(11),D31C(11),T31R(11),T31L(11)
      DIMENSION S23R(11),S23L(11),S23C(11),D23R(11),
     & D23L(11),D21R(11),D21L(11),SPNR(11),SPNL(11),
     & SPNC(11)
      DIMENSION     DR(22,22),DL(22,22),DC(22,22),TR
     &(22,22),TL(22,22),TT(22,22),E(22,22),Q(22,22),
     &S(22,22),DTR(22,22),DTL(22,22),TS(22,22)
      DIMENSION    FEHR(11,9),FEHL(11,9),FEHC(11,9),
     &FEMR(11,9),FEML(11,9),FEMC(11,9),FFR(22,9),FFL
     &(22,9),FFC(22,9),UFO(22,9),UFS(22,9),UFT(22,9)
     & ,DCFR(22,9),DCFL(22,9),DCFC(22,9),FR(22,9),FL
     &(22,9),FC(22,9)
C
  100 READ (1,200) NI,NLC
C
      IF (NI .EQ. 0) GO TO 300
      WRITE (3,210)
      N=2*NI
      NIM1=NI-1
C
C     READ AND WRITE OUT PROPERTIES OF SEGMENTAL ARCHES
C
      READ (1,201) (S11R(I),S33R(I),S13R(I),S23R(I),
     &SPNR(I),S11L(I),S33L(I),S13L(I),S23L(I),SPNL(I),
     &S11C(I),S33C(I),S13C(I),S23C(I),SPNC(I),I=1,NI)
      WRITE (3,211)
      WRITE (3,202) (I,S11R(I),S11L(I),S11C(I),I=1,NI)
      WRITE (3,212)
      WRITE (3,202) (I,S33R(I),S33L(I),S33C(I),I=1,NI)
      WRITE (3,213)
      WRITE (3,202) (I,S13R(I),S13L(I),S13C(I),I=1,NI)
      WRITE (3,214)
      WRITE (3,202) (I,S23R(I),S23L(I),S23C(I),I=1,NI)
      WRITE (3,215)
```

```
      WRITE (3,202) (I,SPNR(I),SPNL(I),SPNC(I),I=1,NI)
C
      DO 105 I=1,NI
C
      S31R(I)=S13R(I)
      S31L(I)=S13L(I)
      S31C(I)=S13C(I)
C
C     SUMS OF VARIOUS STIFFNESSES
C
      SS11(I)=S11L(I)+S11R(I)+S11C(I)
      SS13(I)=S13L(I)+S13R(I)+S13C(I)
      SS31(I)=S31L(I)+S31R(I)+S31C(I)
      SS33(I)=S33L(I)+S33R(I)+S33C(I)
C
      SIG=SS11(I)*SS33(I)-SS13(I)*SS31(I)
C
C     DISTRIBUTION FACTORS OF SEGMENTAL ARCHES
C
      D11R(I)=-(S11R(I)*SS33(I)-S13R(I)*SS31(I))/SIG
      D13R(I)=-(S13R(I)*SS11(I)-S11R(I)*SS13(I))/SIG
      D31R(I)=-(S31R(I)*SS33(I)-S33R(I)*SS31(I))/SIG
      D33R(I)=-(S33R(I)*SS11(I)-S31R(I)*SS13(I))/SIG
      D23R(I)= -S23R(I)*SS11(I)/SIG
      D21R(I)=  S23R(I)*SS31(I)/SIG
C
      D11L(I)=-(S11L(I)*SS33(I)-S13L(I)*SS31(I))/SIG
      D13L(I)=-(S13L(I)*SS11(I)-S11L(I)*SS13(I))/SIG
      D31L(I)=-(S31L(I)*SS33(I)-S33L(I)*SS31(I))/SIG
      D33L(I)=-(S33L(I)*SS11(I)-S31L(I)*SS13(I))/SIG
      D23L(I)= -S23L(I)*SS11(I)/SIG
      D21L(I)=  S23L(I)*SS31(I)/SIG
C
      D11C(I)=-(S11C(I)*SS33(I)-S13C(I)*SS31(I))/SIG
      D13C(I)=-(S13C(I)*SS11(I)-S11C(I)*SS13(I))/SIG
      D31C(I)=-(S31C(I)*SS33(I)-S33C(I)*SS31(I))/SIG
      D33C(I)=-(S33C(I)*SS11(I)-S31C(I)*SS13(I))/SIG
C
C     TRANSMISSION FACTORS OF SEGMENTAL ARCHES
C
      T11R(I)=-D11R(I)
      T13R(I)=-D13R(I)
      T31R(I)=-D31R(I)-D21R(I)*SPNR(I)
      T33R(I)=-D33R(I)-D23R(I)*SPNR(I)
C
      T11L(I)=-D11L(I)
      T13L(I)=-D13L(I)
      T31L(I)=-D31L(I)-D21L(I)*SPNL(I)
  105 T33L(I)=-D33L(I)-D23L(I)*SPNL(I)
      WRITE (3,216)
      WRITE (3,203) (I,D11R(I),D13R(I),D31R(I),D33R(I),
     &I=1,NI)
```

```
      WRITE (3,217)
      WRITE (3,203) (I,D11L(I),D13L(I),D31L(I),D33L(I),
     &I=1,NI)
      WRITE (3,218)
      WRITE (3,203) (I,D11C(I),D13C(I),D31C(I),D33C(I),
     &I=1,NI)
      WRITE (3,219)
      WRITE (3,203) (I,T11R(I),T13R(I),T31R(I),T33R(I),
     &I=1,NI)
      WRITE (3,217)
      WRITE (3,203) (I,T11L(I),T13L(I),T31L(I),T33L(I),
     &I=1,NI)
C
C     DISTRIBUTION MATRICES
C
      DO 110 I=1,N
      DO 110 J=1,N
      DR(I,J)=0.
      DL(I,J)=0.
  110 DC(I,J)=0.
C
      DO 111 I=1,NI
      J=2*I-1
      DR(J,J)=D11R(I)
      DR(J,J+1)=D13R(I)
      DR(J+1,J)=D31R(I)
      DR(J+1,J+1)=D33R(I)
C
      DL(J,J)=D11L(I)
      DL(J,J+1)=D13L(I)
      DL(J+1,J)=D31L(I)
      DL(J+1,J+1)=D33L(I)
C
      DC(J,J)=D11C(I)
      DC(J,J+1)=D13C(I)
      DC(J+1,J)=D31C(I)
  111 DC(J+1,J+1)=D33C(I)
C
C     TRANSMISSION MATRICES
C
      DO 115 I=1,N
      DO 115 J=1,N
      TR(I,J)=0.
  115 TL(I,J)=0.
C
      DO 116 I=1,NIM1
      J=2*I-1
      TR(J+2,J)=T11R(I)
      TR(J+2,J+1)=T13R(I)
      TR(J+3,J)=T31R(I)
  116 TR(J+3,J+1)=T33R(I)
C
```

```
      DO 117 I=2,NI
      J=2*I-1
      TL(J-2,J)=T11L(I)
      TL(J-2,J+1)=T13L(I)
      TL(J-1,J)=T31L(I)
  117 TL(J-1,J+1)=T33L(I)
C
      DO 118 I=1,N
      DO 118 J=1,N
  118 TT(I,J)=TR(I,J)+TL(I,J)
C
      WRITE (3,220)
      DO 119 I=1,N
  119 WRITE (3,204) (TT(I,J),J=1,N)
C
C     DISTRIBUTION AND CARRY-OVER MATRICES
C
      DO 120 I=1,N
      DO 120 J=1,N
      DTR(I,J)=DR(I,J)+TL(I,J)
  120 DTL(I,J)=DL(I,J)+TR(I,J)
      WRITE (3,223)
      DO 121 I=1,N
  121 WRITE (3,204) (DTR(I,J),J=1,N)
      WRITE (3,224)
      DO 122 I=1,N
  122 WRITE (3,204) (DTL(I,J),J=1,N)
      WRITE (3,225)
      DO 123 I=1,N
  123 WRITE (3,204) (DC(I,J),J=1,N)
C
C     MATRIX SUM OF THE INFINITE MATRIX SERIES IN T
C     (INVERSE OF MATRIX Q)
C
      DO 130 I=1,N
      DO 130 J=1,N
      E(I,J)=0.
      E(I,I)=1.
  130 Q(I,J)=E(I,J)-TT(I,J)
      DO 131 I=1,N
      DO 131 J=1,N
      S(I,J)=0.
  131 S(I,I)=1.
      DET=1.
      DO 137 K=1,N
      DET=-DET
      DET=Q(K,K)*DET
      QDIV=Q(K,K)
      DO 132 J=1,N
      Q(K,J)=Q(K,J)/QDIV
  132 S(K,J)=S(K,J)/QDIV
      DO 136 I=1,N
```

```
      QMULT=Q(I,K)
      IF (I-K) 134,136,134
  134 DO 135 J=1,N
      Q(I,J)=Q(I,J)-QMULT*Q(K,J)
  135 S(I,J)=S(I,J)-QMULT*S(K,J)
  136 CONTINUE
  137 CONTINUE
      WRITE (3,221)
      DO 138 I=1,N
  138 WRITE (3,204) (S(I,J),J=1,N)
C
C     PRODUCT OF  T & S
C
      DO 141 I=1,N
      DO 141 J=1,N
      TS(I,J)=0.
      DO 141 L=1,N
  141 TS(I,J)=TS(I,J)+TT(I,L)*S(L,J)
      WRITE (3,222)
      DO 142 I=1,N
  142 WRITE (3,204) (TS(I,J),J=1,N)
C
C     READ AND WRITE OUT EXTERNAL LOADS
C
      DO 150 K=1,NLC
  150 READ (1,205) (FEHR(I,K),FEHL(I,K),FEHC(I,K),
     &FEMR(I,K),FEML(I,K),FEMC(I,K),I=1,NI)
      DO 151 K=1,NLC
      WRITE (3,226) K
  151 WRITE (3,206) (I,FEHR(I,K),FEHL(I,K),FEHC(I,K),
     &FEMR(I,K),FEML(I,K),FEMC(I,K),I=1,NI)
C
C     INITIAL UNBALANCED GENERALIZED FORCES
C
      DO 152 K=1,NLC
      DO 152 I=1,NI
      J=2*I-1
      JJ=2*I
      FFR(J,K)=FEHR(I,K)
      FFL(J,K)=FEHL(I,K)
      FFC(J,K)=FEHC(I,K)
      FFR(JJ,K)=FEMR(I,K)
      FFL(JJ,K)=FEML(I,K)
      FFC(JJ,K)=FEMC(I,K)
      UFO(J,K) =FEHR(I,K)+FEHL(I,K)+FEHC(I,K)
  152 UFO(JJ,K)=FEMR(I,K)+FEML(I,K)+FEMC(I,K)
C
C     SUM OF ALL GENERALIZED FORCES RELAXED AT JOINTS
C
      DO 153 I=1,N
      DO 153 K=1,NLC
      UFS(I,K)=0.
```

```
      DO 153 L=1,N
  153 UFS(I,K)=UFS(I,K)+TS(I,L)*UFO(L,K)
      WRITE (3,227)
      DO 154 I=1,N
  154 WRITE (3,204) (UFS(I,K),K=1,NLC)
C
C     TOTAL UNBALANCED GENERALIZED FORCES
C
      DO 155 K=1,NLC
      DO 155 I=1,N
  155 UFT(I,K)=UFS(I,K)+UFO(I,K)
C
C     DISTRIBUTED AND CARRIED-OVER FORCES
C
      DO 160 I=1,N
      DO 160 K=1,NLC
      DCFR(I,K)=0.
      DCFL(I,K)=0.
      DCFC(I,K)=0.
      DO 160 L=1,N
      DCFR(I,K)=DCFR(I,K)+DTR(I,L)*UFT(L,K)
      DCFL(I,K)=DCFL(I,K)+DTL(I,L)*UFT(L,K)
  160 DCFC(I,K)=DCFC(I,K)+DC(I,L)*UFT(L,K)
C
C     FINAL THRUSTS AND MOMENTS
C
      DO 170 I=1,N
      DO 170 K=1,NLC
      FR(I,K)=FFR(I,K)+DCFR(I,K)
      FL(I,K)=FFL(I,K)+DCFL(I,K)
  170 FC(I,K)=FFC(I,K)+DCFC(I,K)
C
      WRITE (3,228)
      DO 171 I=1,N
  171 WRITE (3,204) (FR(I,K),K=1,NLC)
      WRITE (3,229)
      DO 172 I=1,N
  172 WRITE (3,204) (FL(I,K),K=1,NLC)
      WRITE (3,230)
      DO 173 I=1,N
  173 WRITE (3,204) (FC(I,K),K=1,NLC)
C
C     TEST CONVERGENCE OF TRANSMISSION MATRIX SERIES
C
      CALL EIGEN (TT,N)
C
      GO TO 100
C
  200 FORMAT (2I5)
  201 FORMAT (5F15.9)
  202 FORMAT (I10,3F20.6)
  203 FORMAT (I10,4F15.6)
```

```
  204 FORMAT (10F12.6/(10F12.6))
  205 FORMAT (6F10.4)
  206 FORMAT (I5,6F12.3)
  210 FORMAT (1H1,'                              ',/,10X,
     &'ANALYSIS OF CONTINUOUS CURVILINEAR STRUCTURE',//,
     &10X,'BY GENERALIZED INFINITE MATRIX SERIES METHOD'/
     &////)
  211 FORMAT (//,7X,'JOINT',13X,'R',19X,'L',19X,'C',
     &//,5X,'K11'/)
  212 FORMAT (//,'     K33'/)
  213 FORMAT (//,'     K13'/)
  214 FORMAT (//,'     K23'/)
  215 FORMAT (//,'     SPAN'/)
  216 FORMAT (//,10X,
     &'DISTRIBUTION FACTORS OF SEGMENTAL ARCHES',//,
     &7X,'JOINT',7X,'D11',12X,'D13',12X,'D31',12X,
     &'D33',//,5X,'RIGHT SIDE'/)
  217 FORMAT (//,'     LEFT SIDE'/)
  218 FORMAT (//,5X,'COLUMN SIDE'/)
  219 FORMAT (///,10X,
     &'TRANSMISSION FACTORS OF SEGMENTAL ARCHES',//,
     &7X,'JOINT',7X,'T11',12X,'T13',12X,'T31',12X,
     &'T33',//,5X,'RIGHT SIDE'/)
  220 FORMAT (///,'             TRANSMISSION MATRIX'//)
  221 FORMAT (///,10X,
     &'MATRIX SUM OF THE INFINITE MATRIX SERIES IN T'//)
  222 FORMAT (///,10X,'PRODUCT OF  T & S '//)
  223 FORMAT (///,10X,
     &'RIGHT SIDE DISTRIBUTION-TRANSMISSION MATRIX'//)
  224 FORMAT (///,10X,
     &'LEFT SIDE DISTRIBUTION-TRANSMISSION MATRIX'//)
  225 FORMAT (///,10X,
     &'COLUMN SIDE DISTRIBUTION MATRIX'//)
  226 FORMAT (///,10X,'LOADING CONDITION NUMBER ',I4,
     &//,2X,'JOINT',5X,'FEHR',9X,'FEHL',9X,'FEHC',9X,
     &'FEMR',9X,'FEML',9X,'FEMC'/)
  227 FORMAT (///,10X,'UNBALANCED GENERALIZED FORCES'//)
  228 FORMAT(1H1,10X,'RIGHT SIDE FINAL FORCES'//)
  229 FORMAT(///,10X,'LEFT SIDE FINAL FORCES'//)
  230 FORMAT (//,10X,'COLUMN SIDE FINAL FORCES'//)
C
  300 STOP
      END

      SUBROUTINE EIGEN (A,N)
C
C     THIS SUBROUTINE CALCULATES THE LARGEST EIGENVALUE
C     OF THE TRANSMISSION MATRIX
C
      DIMENSION A(22,22),X(22),Y(22)
      DO 1 I=1,N
    1 X(I)=0.1
```

```
      EIGOLD=1.
      K=1
    2 DO 3 I=1,N
    3 Y(I)=0.
      DO 4 I=1,N
      DO 4 J=1,N
    4 Y(I)=Y(I)+A(I,J)*X(J)
      EIGNEW=Y(1)
      DO 5 I=1,N
    5 X(I)=Y(I)/Y(1)
C
C     TEST CONVERGENCE OF EIGENVALUE
      IF (ABS(EIGNEW-EIGOLD)-0.0001) 8,8,6
C
    6 IF (K-30) 7,9,9
    7 K=K+1
      EIGOLD=EIGNEW
      GO TO 2
    8 WRITE (3,10) EIGNEW
    9 RETURN
   10 FORMAT (//////,10X,'THE LARGEST EIGENVALUE  = ',
     &F7.4//////)
      END

/DATA
```